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ScienceDirect

Computers & Education xxx (2007) xxx–xxx

COMPUTERS & EDUCATION

www.elsevier.com/locate/compedu

Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice

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Received 17 December 2006; received in revised form 29 May 2007; accepted 29 May 2007

Abstract

The idea of 'interpretative flexibility' underpins new approaches to studying technological artefacts and curricular resources in use. This paper opens by reviewing – in this light – the evolving design of dynamic geometry, its pioneering use within classroom projects, and early sketches of its mainstream use in ordinary classrooms. After examining curricular context and its instrumental dimension, the paper then reports a study of teacher constructions of dynamic geometry in classroom practice, conducted in professionally well-regarded mathematics departments in English secondary schools. From departmental focus-group interviews, four teacher-nominated examples of successful practice were selected for study in depth through lesson observation and post-lesson interview. Iterative thematic analysis was employed, first to establish a narrative outline of each case, and then the ideas and issues salient across cases. The study illustrates the interpretative flexibility surrounding the emergent use of dynamic geometry. It found important differences in practical elaboration of the widespread idea of *employing dynamic geometry to support guided discovery*. The process of *evaluating the costs and benefits of student software use* was influenced by the extent to which such use was seen as *providing experience of a mathematical reference model*, and more fundamentally as *promoting mathematically disciplined interaction*. Approaches to *handling apparent mathematical anomalies of software operation* depended on whether these were seen as providing opportunities to develop students' mathematical understanding, in line with a more fundamental pedagogical orientation towards *supporting learning through analysis of mathematical discrepancies.* Such variation was associated with differences in *positioning dynamic geometry in relation to curricular norms* and in *privileging a mathematical register for framing figural properties*. Across all cases, however, *incorporating dynamic manipulation into mathematical discourse* moved implicitly beyond established norms when dragging was used to focus attention on continuous dynamic variation, rather than being treated as an efficient means of generating multiple static figures.

Keywords: Applications in subject areas; Curriculum materials; Dynamic geometry; Educational software; Pedagogical issues; School mathematics; Secondary education; Teacher thinking; Teaching practices; Technology integration

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doi:10.1016/j.compedu.2007.05.013

1. The interpretative flexibility of curricular resources

In the field of social studies of technology, it has come to be recognised that a degree of 'interpretative flexibility' surrounds any technology (Kline & Pinch, 1999). Differing conceptions of the technology come into play not only during the evolution of its central design, but in the course of its propagation as a finished product and its appropriation as a functional tool (Carroll, Howard, Vetere, Peck, & Murphy, 2001; Rabardel & Waern, 2003). Accordingly, the concept of 'innofusion' blurs the conventional technocratic model of development in proposing that aspects of 'innovation' continue throughout the process of 'diffusion' (Fleck, 1988; cited in Williams & Edge, 1996), as a technology and its modalities of use become aligned with user concerns and adapted to use settings (Latour, 1986; cited by Tatnall et al., 1999). This opens the way to variation in modalities of use between user groups and use settings, and to change in modalities of use over time (which may, in turn, precipitate redesign of the material artefact). In this alternative sociocultural model, "design continues in usage" (Rabardel & Bourmaud, 2003, p. 666), and "the conceptualisation of instruments [is] an activity distributed between designers and users" (Rabardel & Waern, 2003, p. 643).

Contemporary educational studies adopt a similar perspective on curriculum materials and pedagogical innovation. The development and diffusion of curriculum materials has long provided a staple approach to influencing classroom practice. However, attempts to 'teacher proof' such materials, and the recurring failure of these efforts even more so, testify that teachers act as interpreters and mediators of curriculum materials (Ball & Cohen, 1996; Haggarty & Pepin, 2002; Remillard, 2005). This reflects a broader pattern in which the unfolding of innovation in education is shaped by the sense-making of the agents involved (Spillane, Reiser, & Reimer, 2002). Teachers typically select from and adapt curriculum materials, and they necessarily incorporate these materials into wider systems of classroom practice, so that the designs of curriculum developers turn out "to be ingredients in – not determinants of – the actual curriculum" (Ball & Cohen, 1996, p. 6).

This paper reports a study of the interpretative flexibility in classroom practice of a particular form of educational software which can be regarded as both technological artefact and curricular resource.

2. The evolving design of dynamic geometry

Although the form of educational software to be studied here has stabilised as 'dynamic geometry', the dynamism which has become its defining characteristic was not such from the start. During the 1980s several pieces of software were developed to provide computer-supported analogues to existing paper-and-pencil processes: to "simulate constructions with a pair of compasses and ruler" and to assist "exact drawing of geometric figures" (Math Forum, undated). The early evolution of *Geometer's Sketchpad* and *Cabri Geometry,* the two products which have become the pre-eminent dynamic geometry systems [DGSs], shows the emergence and increasing prominence of the dynamic feature. *Sketchpad* was originally conceived simply as a program "to draw accurate, static figures from Euclidean geometry" (Scher, 2000, p. 44). In the course of development, however, the idea was adapted from contemporary drawing software of creating a dynamic figure in which points and segments could be dragged while preserving the properties defining the figure. Equally, in the earliest implementation of *Cabri*, it was necessary to access a menu in order to drag; this was not the mode available by default (Scher, 2000, p. 45). And although the dragging operation relatively rapidly became a defining feature of dynamic geometry software, the functional versatility and corresponding complexity of the operation were not anticipated, and have only gradually been charted (Hölzl, 1996; Arzarello, Olivero, Paola, & Robutti, 2002).

Although dynamic geometry systems were developed with educational purposes in view, neither *Cabri* nor *Sketchpad* was initially devised with a particular pedagogical approach in mind (Scher, 2000). However, a significant predecessor, the *Geometric Supposer* (Schwartz, Yerushalmy, & Wilson, 1993), was strongly associated with the idea of providing "a setting and an occasion for conjecturing and creativity for both student and teacher in the mathematics classroom" (Schwartz, 1989, p. 51). A key feature of the *Supposer* was its capacity to generate variant geometrical figures by replicating a construction procedure under differing initial conditions. With the development of dynamic geometry systems, dragging provided a more immediate method of continuous variation of a dynamic figure for similar purposes. Thus pioneering work by mathematics educators quickly associated dynamic geometry with a pedagogical orientation in which such software served "to

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create experimental environments where collaborative learning and student exploration are encouraged'' (Chazan & Yerushalmy, 1995, p. 8 quoted by Balacheff & Kaput, 1996, p. 477). Dynamic geometry systems were seen as ''pedagogic tools finely tuned for the exploration of a mathematical domain'' (Hoyles & Noss, 2003, p. 332) and as ''providing a setting in which students can construct and experiment with geometrical objects and relationships'' (Hoyles & Noss, 2003, p. 333). More strongly still, it was suggested that ''most of the classical problems used in classrooms become obsolete because of the efficiency of [dynamic geometry] environments'' (Balacheff & Kaput, 1996, p. 477).

As these characterisations and claims indicate, computer geometry has been represented as a force for educational change, particularly as part of a wider advocacy of school mathematics reform in the United States. Just as the *Supposer* was represented as ''a lever for change'' (Wilson, 1993, p. 22), proponents of dynamic geometry have suggested its ''potential for changing the beliefs and the behavior of teachers'' in directions which ''reflect a changed view of mathematics, from a static, deductive activity that emphasizes proofs discovered by others to an exploratory, inductive activity that emphasizes the heuristics involved in discovering results'' (Pereira, 2002, p. 2). Correspondingly, it has been suggested that ''dynamic geometry turns mathematics into a laboratory science rather than [a] game of mental gymnastics, dominated by computation and symbolic manipulation'' so that ''mathematics becomes an investigation of interesting phenomena, and the role of the mathematics student becomes that of the scientist'' (Olive, 2002, p. 17 quoted by Gawlick, 2004, p. 1).

3. Computer geometry in pioneering use within classroom projects

Teachers who have worked closely with the designers and advocates of these forms of geometrical software have developed approaches to teaching and learning which reflect this association of the technology with an exploratory pedagogical orientation. More reflexive projects of this type have also highlighted some complexities of translating the envisaged modes of software use into viable forms of classroom practice.

A well-documented early project of this type was carried out in the United States around classroom implementation of use of the *Supposer* (Schwartz et al., 1993). Teachers employed 'laboratory problems' ''designed to lead students to the same abstract geometrical theorems as the more conventional curriculum'', but with the important difference that they were ''designed to have students consider geometrical relationships inductively before being exposed to deductive proof'' (Lampert, 1993, p. 150). This approach had some resonance with ''activities designed to let students discover mathematical theorems empirically'' which teachers already occasionally incorporated into their lessons; for example, having students ''draw intersecting lines and measure the angles that are formed to find that in every case, the opposite angles are equal'' (Lampert, 1993, p. 150). But whereas previously such activities ''supplemented 'teaching as telling' and were done primarily to motivate students and vary the routine'' (Lampert, 1993, p. 150), in the *Supposer* project they were developed into a central element of classroom practice. Operationally this meant that ''students [were to] use the technology to construct and measure many specific figures before... discussion of what the pattern of these measurements imply for relationships in... the universal [figure]'' (Lampert, 1993, p. 150). The major complexities that emerged from this project were those of managing the more differentiated patterns of student thinking and learning associated with a less tightly controlled approach, and of reorganising the sequencing and handling of topics, established within a curriculum framed in deductive terms, so as to accommodate the new element of inductive exploration (Lampert, 1993; Wiske & Houde, 1993).

In a more recent French project developing the classroom use of *Cabri* (Laborde, 2001), teachers devised lesson 'scenarios', and refined them through discussion with a wider team of educators and developers associated with the software. Here, similar complexities of pedagogical management were highlighted, and further complexities of curricular framing. The kinds of 'scenarios' to which the project aspired called for teachers not only ''to accept that they might lose part of their control over what students do'', but ''to accept that learning might take place in computer-based situations without reference to a paper-and-pencil environment'' (Laborde, 2001, p. 311). Nevertheless, in the more conventional types of use developed by some teachers, dynamic geometry provided a convenient parallel to paper and pencil: to produce accurate static figures and generate measurement data; to highlight invariant properties through their visual salience under dragging. Scenarios of these types typically displayed ''a restricted and static use of the possibilities of the software'', and ''the absence of autonomous experimentation by students'' (Laborde, 2001, p. 299). More ambitious scenarios were developed by those experienced teachers who were ''very familiar with the use of technology in mathematics teaching and with research in mathematics education'' (Laborde, 2001, p. 287), if sometimes ''only... under the influence of the researchers of the group'' (Laborde, 2001, p. 302), and subject to it proving possible ''to conceive... interrelations between the new conceptual aspects introduced by the technology and the actual curriculum'' (Laborde, 2001, p. 312). These more adventurous scenarios introduced mathematically innovative uses of dynamic geometry: to pursue qualitatively new types of solution to familiar problems, as with certain aspects of construction; and to pose novel types of problem dependent on the software, such as identifying the properties of a dynamic diagram through dragging it. Even in this development project, then, the ways in which dynamic geometry was used within classroom practice varied considerably with respect to their degree of mathematical innovation and exploratory orientation.

4. Dynamic geometry in mainstream use in ordinary classrooms

These pioneering projects illustrate the ''dialectic between old and new'' (Assude & Gelis, 2002) which teachers encounter in any attempt to integrate innovative resources into school practice. But such projects provide greater support and legitimacy for innovation than is normally available to teachers working in more ordinary circumstances. Indeed, one persistent criticism of advocacy which casts technology as a transformative influence on educational practice has been that this fails to acknowledge ''that patterns of instruction, uses of technology, and treatment of students are heavily influenced by durable organisational structures'' (Cuban, 1989, Cuban, Kirkpatrick, & Peck, 222; Cuban et al., 2001). In particular, advocacy of the *Supposer* (Schwartz, 1989) was censured for casting the software as ''basically a technical fix'' to sustain ''neo-progressive expectations that are out of sync with organisational realities'' (Cuban, 1989, p. 220–221).

Turning, then, to evidence of how dynamic geometry has actually been taken up in schools, the findings of a substantial national survey conducted in the United States (Becker, Ravitz, & Wong, 1999) present an enigma. On the one hand, *Geometer's Sketchpad* was chosen by more than one in five high school mathematics teachers who nominated a 'most valuable' software title for use with their students. On the other hand, this ran counter to other evidence that ''most computer-assigning mathematics teachers use very traditional skill-practice software, [whereas] *Sketchpad* is oriented very differently, towards inductive reasoning and exploration of hypotheses'' (Becker et al., 1999, p. 18). Indeed, later analysis confirmed an association between teachers nominating *Sketchpad* as their most valued software, and reporting skill-development as their main objective for computer use (Becker et al., 1999, p. 46). This suggests that the interpretative flexibility of dynamic geometry is such that its modes of use in mainstream classrooms may differ markedly from the exploratory orientation advocated by many proponents.

In other countries, small-scale studies have offered glimpses of how classroom practices of dynamic geometry use are developing. An English study sketched the approaches of two teachers, apparently working in relative isolation (Lins, 2003). The teachers' concerns about what they saw as the inaccessibility and inflexibility of the software led them to relatively narrow approaches which gave little place to dragging. One teacher emphasised the measuring, tabulating and calculating features of the software to generate data; the other, the hide/show feature of the software to support guess and check activities. This study illustrates two significant aspects of interpretative flexibility: first, neither teacher accorded significance to what the software's designers would regard as its most distinctive functionality; second, each teacher 'privileged' (Kendal et al., 2001) very different software functionalities.

In many mathematics classrooms, a textbook or worksheet scheme provide an important structuring resource for lesson activity (Haggarty & Pepin, 2002; Remillard, 2005). Typically, however, such curriculum materials have not been designed with dynamic geometry in mind, calling for teachers to co-ordinate use of the two resources (or to use each in isolation from the other). A Swedish study found that teachers were concerned to fit their use of dynamic geometry to coverage of the curriculum, employing the software to establish key ideas within a topic before students tackled textbook exercises (Engström, 2004). Although the teachers portrayed dynamic geometry as a tool for students to ''investigate, formulate hypotheses and make conclusions'', the researcher judged classroom enquiry to be tightly structured towards the required textbook results (Engström, 2004, p. 5). This same study also sketched the contrasting case of a Swiss teacher who was more experienced in using the software and able to exercise a higher degree of curricular autonomy. Here the researcher judged classroom activity to conform well to the teacher's representation of dynamic geometry as giving pupils "the possibility to explore, propose, make conjectures and try to demonstrate", with teacher and pupils "work[ing] together... like partners each with their competencies and experiences" (Engström, 2004, p. 4). These contrasting cases illustrate another aspect of interpretative flexibility, in which practical interpretations of investigative use of dynamic geometry are quite different. The contrast between the Swedish and Swiss cases also suggests that these may be conditioned by curricular structures as much as shaped by teacher agency.

5. The curricular context of dynamic geometry use

Much of the pioneering development of dynamic geometry systems has taken place in countries – notably France and the United States – which comparative studies show to have retained a strongly Euclidean spirit within their school geometry curriculum, resulting in greater attention to formalisation and systematisation, including an emphasis on proof (Howson, 1995; Hoyles, Foxman, & Küchemann, 2001). The Euclidean lineage of dynamic geometry might be expected to fit poorly with a national curriculum which refers – as does the English one framing the practice to be studied here – not to *Geometry* but to *Shape, Space and Measures.*

However, the scope to employ the software as a means of supporting observation and measurement resonates with the longstanding orientation of English school mathematics towards "treat[ing] geometry almost entirely as an experimental science, not a deductive one" (Bell, Costello, & Küchemann, 1983, p. 226). In her comparative ethnography of secondary mathematics lessons, Kaiser (2002) notes the predominance in English mathematics classrooms of example-based checking as a means of validating results, not just employed by students, but also promulgated by their teachers. She cites, as exemplary of this trend, a lesson in which one of the 'circle theorems' was established by the teacher setting each student to draw and measure three diagrams to test the result, and then arguing for its acceptance on the grounds that their accumulated checks made it highly plausible.

While empirical approaches to geometry may be particularly pronounced in England, they are also prevalent in many other educational systems. A survey of lower-secondary textbooks from different countries reported that "[geometrical] results are based, in the main, on measurement, observation, and experimentation", supplemented to differing degrees by "exercises in 'local deduction'" (Howson, 1995, pp. 70-71). A broader comparative analysis of secondary-school syllabuses found that many other systems shared the English "emphasis on experimental approaches in geometry and on 'discovering' rather than 'proving' theorems" (Hoyles et al., 2001, p. 3).

In most countries – England included – official curricular specification of the role of ICT in secondary-school geometry has been vague and disconnected (Hoyles et al., 2001; RS/JMC, 2001). Equally, critical review of the dynamic geometry activities proposed in the professional literature suggests that many of these reduce to a verifying approach in which students are simply expected to vary geometric configurations in order to produce empirical confirmation of already formulated results (Hölzl, 2001). Recently, however, the use of dynamic geometry has been given official endorsement and more detailed expression in a government-sponsored elaboration of the English national curriculum at lower-secondary level (DfEE, 2001). The accent of many of the examples is on empirical exploration:

> Use dynamic geometry software to explore properties of lines and circles. For example: Construct a triangle and the perpendicular bisectors of its three sides. Draw the circumcircle (the circle through the three vertices). What happens when the vertices of the triangle move? Observe in particular the position of the centre of the circumcircle (DfEE, 2001, p. 197).

In some other examples, however, such observation of a dynamic figure is presented as a precursor to proving with static diagrams:

> Use dynamic geometry software to construct a triangle with a line through one vertex parallel to the opposite side. Observe the angles as the triangle is changed by dragging any of its vertices. Use this construction or a similar one, to explain using diagrams a proof that the sum of the three angles of a triangle is 180° (DfEE, 2001, p. 183).

Thus, while the treatment of dynamic geometry in this curriculum reflects the continuing influence of an established tradition of experimentation, it also incorporates an aspiration to build from this towards a renewed concern with deduction.

6. The instrumental dimension of dynamic geometry use

The deployment of any tool – such as a manual geometry set or a dynamic geometry package – depends on a cultural-cognitive system of 'instrumental' knowledge; a system concerned not just with immediate manipulation of the tool but with the mediation of action through its use (Artigue, 2002; Rabardel, 2002; Ruthven, 2002). However, the instrumental knowledge associated with a relatively new tool can be expected to be still under development and correspondingly less strongly institutionalised (contributing to the interpretative flexibility which surrounds the tool). Initially, a new technology is likely to be treated as a variant or hybrid of those that are already established and familiar. When a new technology is assimilated to established mathematical methods in this way, it functions as an 'amplifier' of existing forms of action. Only as distinctive affordances of a new technology come to be recognised will it become a 'reorganiser' of mathematical action beyond existing forms (Dörfler, 1993; Pea, 1985).

While the official mathematics curriculum relevant to the study to be reported here explicitly recognises the knowledge required to use manual tools, it overlooks that same aspect of using dynamic software. For instance, the knowledge required to make use of a protractor to measure angles by hand is carefully specified:

> Use a 180° or 360° protractor to measure and draw angles, including reflex angles, to the nearest degree. Recognise that an angle can be measured as a clockwise or anticlockwise rotation and that the direction chosen determines which will be used as the zero line and whether the inner or outer scale will be used (DfEE, 2001, p. 232).

But there is no equivalent attention to the distinctive knowledge required to measure angles with dynamic software: for example, knowing that the measurement tool is applied, not to the line segments forming the arms of an angle, but to the points defining these segments; knowing that, when an angle of interest becomes reflex under dragging, the measurement provided by the software becomes that of the counterpart minor angle. Equally, the curriculum shows little recognition of how dynamic geometry may open up novel mathematical strategies. For instance, the first example cited in the previous section is vague about the stage at which, and the terms in which, a relationship between the perpendicular bisectors of a triangle and the centre of its circumcircle is expected to be established. While, when working with standard manual tools, the construction of the circumcircle depends on constructing the perpendicular bisectors, dynamic software opens up alternative trajectories between triangle and circumcircle. For example, with dynamic geometry, the idea and properties of the circumcircle can be developed in response to one or other (or indeed both) of these problems:

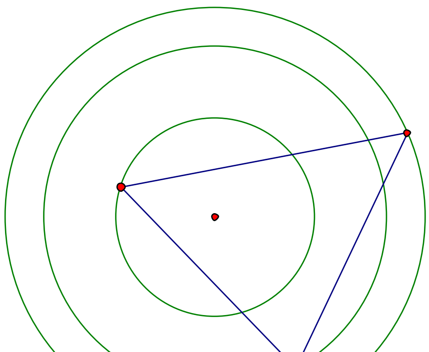

Fig. 1. The idea and properties of the circumcircle of a triangle developed by means of the problem of dragging a common centre so that concentric circles through each of the triangle's vertices coincide.

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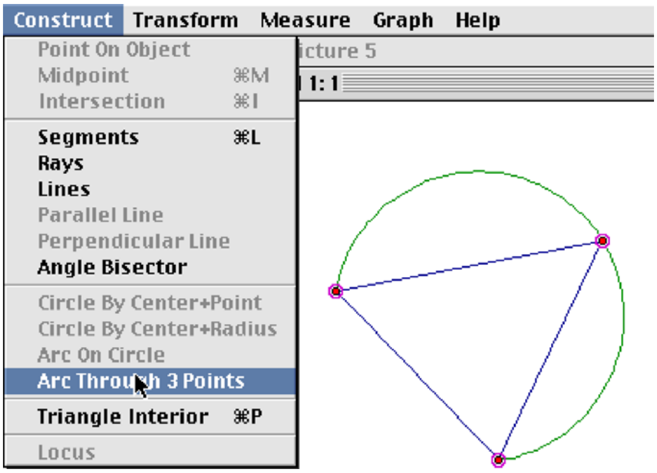

Fig. 2. The idea and properties of the circumcircle of a triangle developed by means of the problem of constructing the unknown centre of an arc through the triangle's vertices.

- Create a free point which acts as the centre of circles through each of the vertices of the triangle (see Fig. 1); by dragging this point and reasoning about its position, identify where any two, and all three of the circles coincide.
- Construct a circular arc directly through the three vertices of the triangle (see Fig. 2); by reasoning about this figure and adding to it, construct the centre of the circle.

  The perpendicular bisectors of the edges of the triangles emerge as key constructs from each of these approaches, but within differing lines of argument.

A curriculum such as this English one, then, overlooks the ways in which techniques of measurement and construction, as well as mathematical strategies available, differ between manual and computer tools.

7. A study of teacher constructions of dynamic geometry in classroom practice

The study of teacher constructions of dynamic geometry in classroom practice which will be reported here arose from a larger research project examining the incorporation of information and communication technologies [ICTs] into the mainstream practice of secondary mathematics and science teaching in England[1].

7.1. Case identification phase of the study: methods and findings

In the first phase of the research, in order to clarify what practitioners themselves regarded as successful practice, and to identify suitable cases for subsequent study in depth, we conducted focus group discussions with school mathematics departments (during the latter part of the 2002/03 school year). We identified departments in state-maintained schools which were regarded as successful, in terms of the general quality of their practice and their use of ICT within it. This was done through crosschecking direct recommendations from professional leaders against relevant information available from sources such as professional publications and inspection reports. In the focus-group interviews, teachers were invited to nominate and describe examples of the successful use of ICT within their teaching. Although official reports based on school inspections indicate that the use of dynamic geometry remains rare in English schools as a whole (OfStEd, 2002, 2004), in 6 of the 11 well-regarded departments consulted by us, examples of the successful use of dynamic geometry were nominated. For the purposes of this study, these sections of the audiotaped discussions were transcribed.

[1] The *Eliciting Situated Expertise in ICT-integrated Mathematics and Science Teaching* project was conducted between 2002 and 2004. Further details are available at <http://www.educ.cam.ac.uk/istl/pub.html>.

These transcripts were first searched to identify references to the kinds of mathematical topic linked to the use of dynamic geometry. The most commonly identified topics were the angle properties of straight line configurations and polygons, and angle properties in circles. All but one department cited at least one of these topics, and half cited both. Also well represented were topics related to classical constructions which were cited by half the departments, mainly in relation to triangles. The transcripts were then analysed to identify key ideas about dynamic geometry expressed across departments (in schools L, M, N, P, Q, R). Through an iterative process of constant comparison and code formation, such ideas were organised thematically. The main clusters of ideas advanced can be summarised in terms of five themes.

Dynamic geometry was valued for *making student work with figures easier, faster and more accurate*, and consequently for *removing drawing demands which distract students from the key point* of a lesson. Various aspects of *making properties apprehensible to students through dynamic manipulation* were expressed. When a dynamic figure was dragged, students could "see it changing" [P] and "see what happens" [L], so that properties "become obvious" [M, N] and students "see them immediately" [N]. In logical terms, ranging across an "infinite number of different scenarios" [N] made it possible to "see that it's always the case and not just that one-off example" [Q], or to "find places where it doesn't work" [R]. By making the key mathematical ideas of a lesson more salient for students, dynamic geometry was seen as supporting teaching approaches based on *guiding students to discover properties for themselves.* What departments variously suggested or implied under this head was that, while teachers might "structure" [P], "hint" [Q], "guide" [N], or "steer" [L] students towards an intended mathematical conclusion, students could largely "find out how it works without us telling them" [P], or "tell you the rule instead of you having to tell them" [R], so that they were "more or less discovering for themselves" [N] and could "feel that they've got ownership of what's going on" [P]. Finally, using dynamic geometry in these ways was seen as *promoting student conviction, understanding and remembering,* in terms of the depth of student conviction about results [M, N], understanding of topics [L, P], and remembering of material [N, Q, R].

7.2. Case analysis phase of the research: design

In the second phase of the research, we sought greater insight into a range of practice, through lesson observations and post-lesson interviews (conducted during the earlier part of the 2003/04 school year).

Available resources made it feasible to follow up at most four cases, and so priorities had to be decided. It was clearly important to examine the most commonly mentioned topic: that of angle properties; in addition, information from the first phase suggested some differences of approach to this topic. Equally, it was desirable to include other topics, particularly some of those where the available information suggested distinctive concerns and approaches. Within these constraints, it proved possible to achieve a purposive sample satisfying the priorities noted above by pursuing the main example of successful dynamic geometry use nominated by each of the teachers who had proved more informative in discussing their use of dynamic geometry. While a study on a larger scale and with a different design would clearly be required to characterize the full variety of classroom practice associated with dynamic geometry use, this set of four cases promised to capture important aspects of that variation.

Post-lesson interviews were conducted after each observed session. These were organised around a standard sequence of printed cards asking teachers about their thoughts, first while preparing the lesson (what they wanted pupils to learn; how they expected use of the technology to help pupil learning); then looking back on the lesson (how well pupils learned; how well the technology helped pupil learning; the important things that they were giving attention to and doing); followed by further related prompts. Finally teachers were asked for their thoughts about pitfalls of dynamic geometry (for which the most productive prompts concerned the main pitfalls they had experienced; and ways they had found of avoiding or managing these pitfalls). These interviews were audiotaped and transcribed. The subsequent process of analysis was in two stages.

7.3. Case analysis phase of the study: case outlines

The first stage of analysis was within-case and more emic in approach, seeking to encapsulate the teachers' accounts of their practice. The interview transcripts for each case, informed where relevant by the lesson

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observations, were analysed through an iterative process of constant comparison and code formation. The aim was to identify the main themes informing thinking about the practice in question, constituting its guiding 'practical theory' (Deaney, Ruthven, & Hennessy, 2006). These themes were then organised, and illustrative transcript segments selected, to create a narrative outline of the case. The case outlines are presented in Tables 1–4 under headings designed to identify their distinctive characteristics as a teacher construction of dynamic geometry in classroom practice, and with labels indicating which school (N, P, Q) they came from:

- Case N: Prepared teacher use of a dynamic figure to support persuasive mathematical presentation (Table 1).
- Case P1: Structured student use of dynamic geometry to support mathematically principled interaction (Table 2).
- Case P2: Structured student use of a customised dynamic figure to provide a mathematical reference model (Table 3).
- Case Q: Guided student use of dynamic geometry to support mathematically disciplined expression (Table 4).

7.4. Case analysis phase of the study: salient issues

The second stage of analysis was cross-case and more etic in approach; seeking to identify and conceptualise salient issues across all four cases, informed by the earlier literature review. Whereas the more decontextualised focus-group accounts of successful technology use had tended to accentuate shared commonplaces, the case-study material – more immediately grounded in concrete episodes of technology use – served better to bring out complexities unacknowledged in the earlier accounts, and to signal contrasting responses to them. In the presentation that follows, each issue is characterised, and related to particular cases, often employing short citations from case outlines.

7.5. Employing dynamic geometry to support guided discovery

The departmental interviews indicated that dynamic geometry was valued for the contribution it could make to guiding students to discover mathematical properties for themselves. The case studies illustrate a range of practical expressions of this idea. One case involved a strongly teacher-led, whole-class approach, in which "the technology helped a great deal" by making it easier for students to "spot the rule", and thus for the teacher "to get the correct response out of them", so that "you're not just telling them a fact, you're allowing them to sort of deduce it and interact with what's going on" [N]. In the other cases, the classroom approaches involved more delegation to students, through 'investigations' structured towards similarly preconceived mathematical results, with the teaching role important in "drawing attention to" [P1], "flagging up" [P2] and "prompting" [Q] these results, as well as in helping students to surmount difficulties encountered.

7.6. Evaluating the costs and benefits of student software use

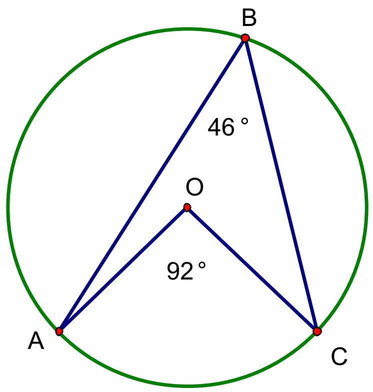
The case records contain allusions to what the departmental interviews had portrayed as the benefits of dynamic geometry – compared to established manual tools – in facilitating work with figures. However, they also reveal important differences of perspective on student software use, leading to approaches based on *avoiding, making manageable,* or *accepting* its demands. In three of the cases dynamic geometry was viewed as "a difficult program for the students to master" with "huge scope for... mistakes and errors" [N], "quite a difficult piece of software" which "does add complications" [P1], and a package with "peculiarities [and] foibles [which] are quite difficult for pupils to... understand and to overcome" [P2]. In one of these cases, then, the software was used only for teacher presentation on the grounds that "it would take a long time... for [students] to master the package" and "the return from the time investment... would be fairly small", so that "the cost benefit doesn't pay" [N]. In those other cases where concerns about the accessibility of the software were expressed, the normal pattern was "to structure the work, so [that students]... don't have to be

Table 1
Case N outline: Prepared teacher use of a dynamic figure to support persuasive mathematical presentation

This example, concerning angle properties in the circle, was nominated in these terms:

> "The one that I do like to do is the one with the circle theorem that says the angle at the centre is twice the angle at the circumference, because that covers the same theorem as the angles in the same segment, and the angle at the semi-circle is 90 degrees, and you can cover a lot of different circle theorems by doing that one demonstration. And the students find it very, very difficult to believe if they don't see it on the computer... And yet when you're dragging it round this circle using Cabri... it just gets that they start to believe a lot more and they are more convinced of its truth."

We observed a lesson that lasted for one 40-minute session, and which involved a Year 10 set of students (aged 14–15) of below-average academic attainment. [We also studied a further single-session lesson with another class, on angle sums of polygons, which displayed very similar features.]. The session started with 25 minutes of activity led by the teacher, during which he constructed and manipulated a dynamic figure, projected from his computer onto an ordinary whiteboard. This was followed by individual student activity on a (pencil and paper) textbook exercise.



Maintaining student attention through dynamic presentation and tactical questioning
Dynamic geometry presentation was valued for holding the attention of students:

> "I think that holds their attention more. The fact that they can see that you can pick up and drag these shapes around, and then the angles change as well automatically, so all the numbers are changing... That sort of movement, that dynamism, helps to keep their attention."

Tactical questioning of the class was designed to direct students towards a target result:

> "[I] pick on students that I think may have a problem with it... If students that I think are going to have a problem with it understand it, then I can be fairly confident that the others understand it as well... If [I] ask a student a question and they don't know the answer, I won't give up on them, I'll carry on asking or trying to get the correct response out of them... Not only is it benefiting the student you've asked, everybody else is trying to come to the correct solution as well. And so I do that to help reinforce what's being presented."

Making properties apprehensible and convincing to students through dragging
Dragging figures was seen as a powerful means of making properties apprehensible and convincing to students, notably the unchanging measure of the angle at the circumference:

> "If you do it on the board and you drag the thing round, then they tend to be much more convinced by what they see. So, I think the technology helps because they can actually see it getting dragged round, they see the angle doesn't change and they are much more convinced."

Making the relationship between the angle at the circumference and the angle at the centre derivable by students depended on their identifying a pattern as the pairs of measures varied under dragging:

> "Trying to get across the point that the angle at the centre is twice the angle at the outside... Because the angle automatically changes as you drag the point round, you can write up pairs of values, [and] the students can deduce that themselves... So the technology helps a great deal in that respect because you're not just telling them a fact, you're allowing them to sort of deduce it and interact with what's going on."

Table 1 (*continued*)

Making it easy for students to identify properties by pre-empting possible confusions
Great care was taken to anticipate and pre-empt situations which might confuse students:

> "[I] keep things running through the lesson in my own head, and looking for possibilities where students may become confused, or things that might cloud the issue, so that I can do something about that before it becomes an issue in the classroom."

Dynamic figures were designed and manipulated in ways intended to make target properties as readily discernible by students as possible:

> "I obviously pre-prepared the circle with the lines and angles already marked in. Also for this group, I made sure the angles were always integer values... That way you don't have half angles to deal with. So the angle at the centre was always an even number of degrees because that way the angle at the outside can be halved quite successfully... So I did that to help make it a little bit easier for them to spot the rule."

In particular, the teacher avoided dragging figures into positions where any angle would become reflex, resulting in its smaller counterpart angle being measured by the software:

> "When you move things around, if the three points you are measuring swap over somehow, then it starts measuring a different angle."

Avoiding the disadvantages of student software use through teacher presentation
Software use was limited to teacher presentation to avoid the demands and difficulties of students themselves using it:

> "If I wanted the students to do it, it, it would take a long time in order for them to master the package and I think the cost benefit doesn't pay there... And there's huge scope for them making mistakes and errors, especially at this level of student."

The educational returns from students learning to use the software were seen as insufficient given a curriculum which was more factual than investigative, and in which relatively few topics benefited from dynamic geometry treatment:

> "It's a difficult program for the students to master... The return from the time investment... would be fairly small... And the content of geometry at foundation and intermediate level just doesn't require that degree of investigation. So they need to learn certain facts... but most of those facts can be learned quite well enough without Cabri."

Table 2
Case P1 outline: Structured student use of dynamic geometry to support mathematically principled interaction

This example, concerning angle properties in various configurations, was nominated in these terms:

> "All of our angle work at [lower secondary] is done [with DGS]... Most of the tasks are... designed with what we want to achieve from it in mind. So if we want them to see that the angles on a straight line add to 180 it's designed exactly for that purpose... So they should come to the right conclusion but... they feel that they've done it on their own and they've explained it."

We observed two lessons, each lasting for one 50-minute session. We will focus on a lesson on angle sums of polygons, which involved a Year 7 set of students (aged 11-12) of average to above attainment, in their first year of secondary education. The session started with 15 minutes of activity led by the teacher, using an interactive whiteboard. This was followed by the students working in trios at a computer, guided by a teacher-devised worksheet, to investigate the angle sums of dynamic polygons that they themselves constructed. [A second lesson with another class was disrupted by technical difficulties.]

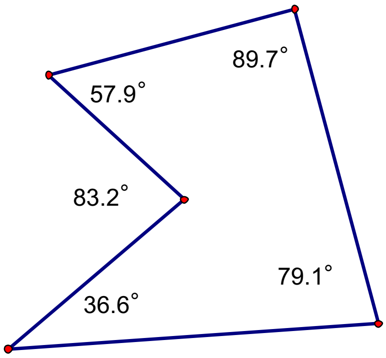


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Table 2 (*continued*)

Developing student awareness of space and shape through exploring dynamic figures

The emphasis of the lesson was more on promoting students' broad understanding of shape and space than their knowledge of specific results:

> "The work on angles in polygons is not so important; that is extension work, and we do it again, that topic, later in the year... So it's sort of a lead up for that as well. But in terms of key concepts, it's not really the curriculum topic that's important, but the understanding of space and shape in geometry, and how that works."

While other investigations aimed at establishing specific results were more structured, this lesson was concerned with developing broader spatial awareness, and so was more fluid:

> "[This] lesson was more about them getting to know the software, them having an awareness of space, and thinking how shapes grow and what happens to some of the corners as they grow, and whether that's related to the shape. And that was much more fluid... I was a bit more adding on to the end of a topic, and you can be a bit wilder there, and not have to follow the curriculum as such."

Giving students experience of geometrically principled interaction with the software

An important feature of DGS software was seen as being the way in which its design around geometrical principles shaped student interaction with it:

> "The package is geometry-based, and it is from-first-principles geometry... One of the main parts of this lesson was that they could learn the software, and have some idea of how shapes and points relate to each other, and to see that the software works geometrically."

In particular, the teacher could build on students' experience of making use of the software, to draw out the way in which they had been enacting geometrical principles:

> "When they were trying to measure the angle, that really brought out the idea of what is an angle... Just the action of doing it really made a fuss about that for them, and they really understood that angles, these three points that are on two lines, and what it means."

Focusing student attention on mathematical essentials through structured software use

It was seen as important to structure students' use of the software around the dragging of simple figures in order to focus on mathematical essentials:

> "It does add complications, because it's quite a difficult piece of software. So that's why we structure the work so they just have to move points. So they don't have to be complicated by that, they really can just focus on what's happening mathematically."

The ease with which figures could simply be dragged to create multiple examples was seen as contributing to achieving this focus:

> "As they move the shapes around, they can see what's actually happening, as if they were drawing it on a page, and doing different drawings. But it obviously removes the need for them to have to redraw things. So it's easier for the kids who find it hard to draw well. So it really helps them, and they can focus on the learning of how the angles match, and what they add up to."

Supporting students in questioning unexpected results and learning from them

Supporting students in identifying and analysing the sometimes confusing way in which the software measured angles was intended to contribute to their learning:

> "I wanted to draw attention to... how the software measures the smaller angle, thus reinforcing that there are two angles at a point and they needed to work out the other... Because a lot of them had found that they'd got the wrong answers, and [that] it measured the obtuse angle rather than the reflex angle, so I highlighted that, because that was important in terms of understanding the software. Next lesson, we'll talk a lot more about what we learned from it."

Anomalous situations of this type were valued for developing students' critical mathematical thinking about results produced by the software:

> "For me, success is when the kids produce something and then say "This can't be right because it's not what I expect"... Because they're going to make mistakes. But if they look at it... they can sense that there's something wrong... So we talked about how we'd overcome that... I think that happened in slightly different ways around the room, but it was one of the key things that the kids learned, that you can't assume that what you've got in front of you is actually what you want, and you have to look at it... and question it, which is very powerful."

This reflected a wider emphasis on supporting students in thinking through conflicting states of affairs to a coherent resolution:

> "Where there is a conflict like that and the child's not understood something or finds there's something not right, I question them about what's not right and why it's not right, and therefore what do they think it should be?... All the time [I'm] subconsciously thinking, what will challenge this child, what will open the door for them to take this step through?"

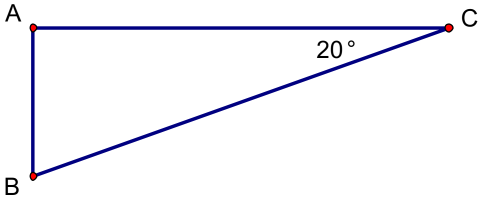
Please cite this article in press as: Ruthven, K. et al, Constructions of dynamic geometry: A study ..., *Computers & Education* (2007), doi:10.1016/j.compedu.2007.05.013

Table 3
Case P2 outline: Structured student use of a customised dynamic figure to provide a mathematical reference model

This example, concerning the trigonometry of right-angled triangles, was nominated in these terms:

> "I think their understanding of trigonometry was tremendously deeper through using the idea of being able to vary the triangles, and understand that if they extended the triangle the angles stayed the same, the ratio stayed the same... They could see... that there was a definite connection between the angle and the trig ratio that was coming up, and that really seemed to empower them."

We observed the first and last of six 50-minute sessions on this unit of work with a Year 11 set of students (aged 15-16) of broadly average academic attainment. Each session started with 5-10 minutes of class activity led by the teacher, using an interactive whiteboard. This was followed by an extended period of student activity, working in pairs at a computer, guided by a teacher-devised worksheet. Each session concluded with a 2-minute period of teacher-led class summary. Throughout the unit, a customised figure – subject to hidden constraints – provided a dynamic model of a right-angled triangle. Dragging one vertex (B) altered the size of the displayed angle; dragging another (C) scaled the whole triangle up or down.



Building student understanding through paired investigation of the dynamic model
The idea of building student understanding through investigating variation in the customised figure guided the opening session. Students examined triangles of differing size, tabulating the lengths and ratio of opposite and adjacent edges, and seeking patterns:

> "The first lesson was very much investigative. There's an element of surprise with pupils... that there is this relationship between the sides... It's not intuitive that there is a link between the ratios."

Compared to working by hand, using a dynamic figure was seen as leading to students developing a better overview, through making the process of data generation more coherent, and bringing out the covariation of values under dragging:

> "If they were doing that same lesson without the technology, I would expect them to subdivide the number of triangles up between them. But... the results could almost be gathered in isolation, and although they would be brought together, there's no sense of how as the triangles grow, the results grow. And I think that's important, that they can actually see as they move the triangle that result for the tan growing."

Pairwork was intended to stimulate peer discussion at a level proximal for students' development:

> "Where pupils discuss ideas with each other... their language and the gap between their knowledge is smaller than the gap between theirs and mine, so they're actually learning better from each other, rather than learning from me... They use the computer as a mediator really, and it helps the discussion along."

Prompting students to appreciate errors by relating solutions back to the dynamic model
The customised figure served as a continuing reference model throughout the topic:

> "[Students can] use... an idea which they've developed right from the basics of linking the three sides together, to come up with these three ratios, and seeing the connections between different triangles... And I think that really goes in and helps them understand the process of using trigonometry to... work out the missing sides. I think the thing that makes the learning successful is that it actually comes from them rather than me."

In the last session, students tackled problems working out a missing length in a right-angled triangle through creating and solving an appropriate trigonometric equation; they then checked their answer by dragging the customised figure until the displayed measures matched the problem. This was seen as not just alerting students to any error, but as prompting them to relate their answer back to the geometric situation:

> "Often when pupils sit down and do trigonometry questions, they've got no idea whether the answer they're coming up with is actually right or wrong... A very common misconception... [is that] they'll work out the length of the hypotenuse and get a shorter length than they should, and not appreciate that they're wrong. Whereas [in the lesson] I saw examples of where they understood that their answers were wrong, and that they needed to adapt their method in some way."

Making software use manageable for students by preparing figures and measurements
Providing students with a prepared figure was judged preferable to getting them to construct one for themselves, because of complexities of the software:

(*continued on next page*)

Table 3 (*continued*)

> "[If] the pupils have to create the diagram themselves, [that] can be quite a negative experience because some of the peculiarities, the foibles, with things like Cabri are quite difficult for pupils to... understand and to overcome."

Similar concerns about manageability for students led to a borrowed worksheet being adapted to ensure that all triangles were of a size which would fit within the normal screen display:

> "I modified the worksheet... so that they could get all the measurements on the screen... If I'd had to get the pupils to think about... scaling it down so they could get it onto the screen, that that would have complicated things. So it was making it manageable for them."

The customised dynamic figure was designed to afford just those types of variation required:

> "The file was set up so that there were restraints on how the triangle moved. So that it could be enlarged, or the angle could change, but always with a fixed right angle... Something that they could use to generate all their results for them."

Bringing out key points through guiding student use of the customised figure

The customised dragging to scale the figure was informally mathematised as an enlargement:

> "Going through this unit of work in this way, I think helps most pupils understand that we're looking at enlargements... that they can then develop that idea from an enlargement, which is quite a concrete idea for most pupils, into something far more abstract."

The teacher played an important part in priming student understanding of the distinctive types of variation which dragging particular vertices was intended to provide:

> "Making it clear about which points would move on the triangle, to vary length and vary angles, I think, was a key point, so that they could actually understand what was going on there."

Indeed, such teacher guidance of the use of the customised figure played an important part in conveying the key mathematical ideas of the lesson:

> "I think if I'd not flagged that particular point up...that they wouldn't have realised that the angle was constant, or the result was constant, and they... would have missed that key point."

Table 4
Case Q outline: Guided student use of dynamic geometry to support mathematically disciplined expression

This example, concerning the idea of the 'centre' of a triangle, was nominated in these terms:

> "We'd done some very rough work on constructions with compasses and bisecting triangles. And then I extended that to Geometer's Sketchpad on the interactive whiteboard... And we... bisected the sides of a triangle. And [the pupils] noted that [the perpendicular bisectors] all met at a point... And we moved it around and it wasn't the centre of the triangle. Sometimes it was inside the triangle and sometimes outside... So we had... the whole lesson, just discussing what's the centre of a triangle."

We observed a lesson over two 45-minute sessions on consecutive days, which involved a Year 7 class of students (aged 11-12) in their first year of secondary education. Both sessions started with 10-20 minutes of teacher-led activity, using an interactive whiteboard. This was followed by student activity at individual computers, guided by a teacher-devised worksheet. In the first session, students themselves constructed a dynamic triangle and the perpendicular bisectors of its edges, in order to investigate the properties of the bisectors. In the second session, they investigated how changing the shape of the triangle affected the position of the point of concurrence.

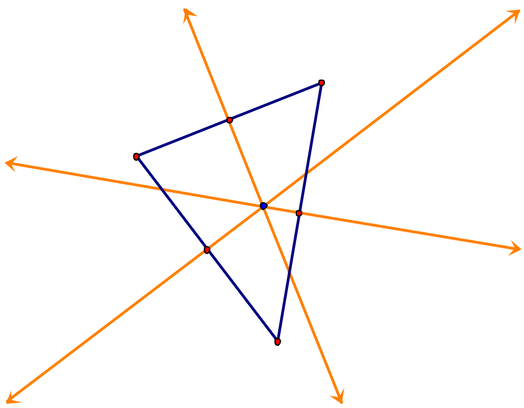


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Table 4 (*continued*)

Giving students experience of finding rules and patterns within abstract geometry

The lesson was intended to follow on from work on construction by hand, extending this beyond the official curriculum:

> "Geometry's so vague in school maths at the moment... I mean the traditional national curriculum way would be to do the construction, and then on loci and stuff like that... [But] I went on more of the geometry way."

This involved learning to explore the properties of familiar shapes and conceptualise them in more abstract geometric terms:

> "The main thing is the idea that you can look at a shape that you're fairly familiar with and do things to it, and find new ideas. It's this idea of coming across abstract geometry and finding rules and patterns within it, which is what geometry's all about really."

The focus was more on experiencing geometrical exploration than on mastering particular content:

> "It's less about learning the actual facts, than about the ideas of exploring an abstract geometrical idea, and finding that there's lots of different rules... They will have manipulated shapes, they'll have got used to... the idea of trying to describe things. There's a lot of maths there that isn't directly learning facts, and I think that's a really important part of doing that sort of exercise."

Emphasising mathematical rules through clarifying student instructions to the computer

The way in which the software required clear instructions to be given in mathematical terms was seen as a key characteristic:

> "I always introduce Geometer's Sketchpad by saying "It's a very specific, you've got to tell it. It's not just drawing, it's drawing using mathematical rules."... They're quite happy with that notion of... the computer only following certain clear instructions."

As students worked on constructions, the teacher helped them to identify mistakes and analyse them mathematically:

> "[Named student] had a mid-point of one line selected and... a perpendicular line to another, and he didn't actually notice... When I was going round to individuals; they were saying "Oh, something's wrong"; so I was [saying] "Which line is perpendicular to that one?""

Such difficulties were viewed as beneficial in drawing out the mathematical ideas at stake:

> "A few people... drew random lines... because one of the awkward things about it is the selection tool... [But] quite a few discussions I had with them emphasised which line is perpendicular to that edge... So sometimes the mistakes actually helped."

Making mathematical properties noticeable through prompting dragging by students

The teacher noted the crucial part he played in making key mathematical properties remarkable by prompting students to drag the vertices of the dynamic triangle to bring out the concurrence of the perpendicular bisectors:

> "They didn't spot that they all met at a point as easily. I think it just doesn't strike them as being particularly unusual... I don't think anybody got that without some sort of prompting. It's not that they didn't notice it, but they didn't see it as a significant thing to look for."

Similar prompting of dragging helped students to appreciate how the position of the point of concurrence – inside or outside the triangle – could be related to the size of the largest angle of the triangle – acute or obtuse:

> "It was nice to see the way that the point, the central point, went from inside to outside. They were able to move that around and look how the angle was changing. And what sort of rules... I led them very closely on that."

Making learning less vague through getting students to write a rule clearly

Formulating findings in explicit mathematical terms was seen as an important issue:

> "They've got to actually write down what they think they've learned. Because at the moment, I suspect they've got vague notions of what they've learnt but nothing concrete in their heads."

Accordingly, the teacher sought to sharpen the precision with which students expressed their conclusions:

> "I was focusing on getting them to write a rule clearly. I mean there were a lot writing "They all meet" or even, someone said "They all have a centre"... So we were trying to discuss what 'all' meant, and a girl at the back had "The perpendicular bisectors meet", but I think she'd heard me say that to someone else, and changed it herself; "Meet at a point"."

This refining of mathematical expression was assisted by the provisionality of the text accompanying figures:

> "The fact that they had a text-box... and they could change it and edit it, they could actually then think about what they were writing, how they describe. I could have those discussions. With handwritten, if someone writes a whole sentence next to a neat diagram and you say..."Can you add this in?", you've just ruined their work. But with technology you can just change it, highlight it and add on an extra bit, and they don't mind."

complicated by that [and]...can just focus on what's happening mathematically'' [P1], which could extend to careful customisation of a prepared figure and accompanying worksheet by the teacher to ''mak[e] it manageable for them'' [P2]. Such concerns were not expressed in the final case [Q], perhaps because it involved student use of the software being exploited more fully as a vehicle for developing and disciplining geometrical thinking. Indeed, the use of dynamic geometry by students themselves occurred in exactly those cases where such use was seen as contributing directly to building mathematical knowledge, as will be explained more fully under the following themes.

7.7. Handling apparent mathematical anomalies of software operation

A further challenge for teachers was in handling the apparent mathematical anomalies of software operation which arise when figures are dragged to positions where an angle becomes reflex (with the associated problem of measurement), or where rounded values obscure an arithmetical relationship between measures. In one case [N], great care was taken to avoid exposing students to such anomalies, through vigilant dragging by the teacher. The other case tackling the same type of topic (and indeed an identical topic in one lesson) provided a striking contrast. The lesson actually sought ''to draw attention to... how the software measures the smaller angle'' so that the apparent anomaly of measurement for reflex angles could be resolved through mathematisation, ''thus reinforcing that there are two angles at a point and [that students] needed to work out the other'' [P1]. Moreover, in this case, ''one of the key things that the kids learned'' was ''that you can't assume that what you've got in front of you is actually what you want, and you have to look at it... and question it'' [P1]. These strategies differed, then, between – in the first case – *concealing* anomalies of software operation and – in the second case – *capitalising* on them for purposes of mathematical knowledge building.

7.8. Supporting learning through analysis of mathematical discrepancies

Capitalising on such anomalies formed part of a wider teaching strategy in which ''where there is a conflict like that and the child's not understood something or finds there's something not right'', the preferred response was to ''question them about what's not right and why it's not right'' [P1]. The two further cases in which students worked with the software display comparable emphases on exploiting such errors and anomalies so as to promote mathematical thinking and knowledge building. Where students encountered difficulties in constructing figures correctly with the software, it was suggested that ''sometimes the mistakes actually helped'' by leading to discussions which drew out the mathematical ideas at stake [Q]. Likewise, where students checked their analytic solutions against a reference figure provided by the software, this was seen not just as alerting students to error, but as prompting them to relate their answer back to the geometric situation [P2]. In the remaining – and contrasting – case, one of the reasons offered for not having students themselves work with the software was the ''huge scope for them making mistakes and errors''; as already noted, teacher manipulation of the software sought to avoid student exposure to anomalies. While classroom questioning in this case was intended to ''pick on students that... may have a problem'' and ''carry on... trying to get the correct response out of them'', this was presented as a means more of managing lesson pace and maintaining student attention than of encouraging rethinking [N].

7.9. Promoting mathematically disciplined interaction through the software

A second aspect of mathematical knowledge building through software use is highlighted in those cases where students were tasked to construct dynamic figures. Here, the mathematical discipline promoted by student interaction with the software was stressed. Experiencing ''from-first-principles geometry'' was intended to help students ''see that the software works geometrically'', and to give them ''some idea of how shapes and points relate to each other'' [P1]. Likewise, the ideas of ''the computer only following certain clear instructions'' and of the software ''not just drawing [but] drawing using mathematical rules'' were made explicit to students [Q]. And there was a further emphasis on getting students to formulate their conclusions as a mathematically precise sentence, using the software ''text-box [to] change it and edit it... [so as to] actually then think about... how [to] describe'' [Q].

7.10. Providing experience of a mathematical reference model through the software

The teaching approach in both these cases [P1, Q] can be viewed as exploiting students' interaction with the software to give them experience of a standardised reference model for geometry, generalisable across situations. Such interaction was rather different in the other case where students worked with the software [P2]. There, a concern to make work ''manageable'' for students led to the use of a customised figure prepared by the teacher to provide a more local reference model for a particular topic. Rather than employing the mathematically standard enlargement operation provided by the software, this customised figure used hidden relations to fashion an informal local substitute operated simply by dragging a particular point of the figure. However, this necessitated special guidance to students to ''mak[e] it clear about which points would move on the triangle, to vary length and vary angles... so that they could actually understand what was going on there'' [P2]. Thus, while use of the customised figure might be efficient in the immediate context of this unit of work, the non-standard relation that it established between enlargement and dragging introduced a degree of dissociation between this local reference model for trigonometry and the more global reference model for geometry provided by the software as a system.

7.11. Positioning dynamic geometry in relation to curricular norms

The contrasts which have been noted can be seen as positioning dynamic geometry differentially in relation to curricular norms. In both the cases [P1, Q] which sought to promote mathematically disciplined interaction through the software, teachers talked of the lesson as doing more than just following the national curriculum (although both these lessons matched suggestions to be found in the official elaboration of the national curriculum). In one case, this was seen as a matter of the lesson ''adding on to the end of a topic'' so that ''you can be a bit wilder there, and not have to follow the curriculum as such''; in particular, ''it [was] not really the curriculum topic that [was] important, but the understanding of space and shape in geometry'' [P1]. In the other case, a concern that ''geometry's so vague in school maths at the moment'' motivated a lesson which sought to develop earlier work, not in ''the traditional national curriculum way'', but ''more of the geometry way'', to give students experience of ''coming across... abstract geometry and finding rules and patterns within it, which is what geometry's all about really'' [Q]. Both of these cases involved younger classes (both in their first year of secondary education). In contrast, use of the software was more tightly structured in those cases involving older classes (correspondingly closer to national tests and examinations), and the lessons were more sharply focused on the assessed curriculum. In one, students worked with prepared figures carefully tailored to assist them in developing an understanding of assessed curriculum objectives [P2]. In the other, the lesson was similarly focused on specified objectives, with the curriculum viewed as ''just [not] requir[ing] that degree of investigation'' which would call for student use of the software [N].

7.12. Privileging a mathematical register for framing figural properties

As already noted, one case [Q] is distinctive in going beyond the official curriculum to a more classical emphasis on use of a *geometrical register* to frame and analyse figural properties. A concern that students should have experience of ''finding rules and patterns'' in ''abstract geometry'' led to them being asked to ''write a rule clearly'' as a means of ''trying to describe things'' in directly geometrical terms [Q]. By contrast, in the other three cases, figural properties were inferred less directly from arithmetical patterns in the numerical measures generated by a dynamic figure. Because ''you can pick up and drag these shapes around... so all the numbers are changing'', students could ''see [that] the angle doesn't change'', or use ''pairs of values... [to] deduce [for] themselves'' the relationship between two angles [N]. By using a dynamic figure ''to generate all their results'' [P2] in this more familiar *arithmetical register* – privileged by an official curriculum placing emphasis on the development of numeracy – students could ''focus on the learning of how the angles match, and what they add up to'' [P1].

7.13. Incorporating dynamic manipulation into mathematical discourse

Finally, the case records show how mathematical discourse was developing to incorporate dynamic manipulation. Quite often, dragging was presented simply in terms of moving or changing a figure to generate discrete static examples: "move the triangle around and try different triangles within seconds" [Q]; "change it and you've got then an unlimited number of shapes" [P1]. However, dragging was sometimes framed in more dynamic terms. Working in the directly geometrical register, for example, attention was drawn to "the way that the point, the central point, went from inside to outside [the triangle]" so that students "were able to move that around and look how the angle was changing" [Q]. Working in the more prevalent arithmetic register, attention was focused more sharply on the constancy, variation and covariation of measures across a domain; for example, where students "can actually see as they move the triangle that result... growing" [P2]; or "can actually see it getting dragged round, they see the angle doesn't change" [N]. This last example, from the 'circle theorems' lesson, was particularly interesting because it led on to a classroom episode which tacitly bridged between dynamic and static versions of the 'same' mathematical result. In this episode, the dynamic image of a single moving angle-at-the-circumference (as shown in the figure in Table 1) was related to the static image of two fixed angles-at-the-circumference (characteristic of the textbook examples which students were to tackle later in the lesson), through students being invited to predict the result of dragging the single angle in the dynamic figure to a new position already superimposed by hand to form a fixed second angle within the projected image.

8. Concluding summary and discussion

The idea of interpretative flexibility acknowledges, first, that ideas about a form of software develop during the process of its design and pioneering use. Under development, dragging became a progressively more central and salient feature in the design of geometry software, as the mathematical potential of dynamic manipulation became recognised. Likewise through pioneering use of dynamic geometry, it came to be recognised that dragging had much greater functional versatility than had been anticipated. Wider diffusion of software introduces a second aspect of interpretative flexibility. While it is relatively straightforward to distribute copies of a software artefact, the ideas and methods associated with its use cannot be replicated so easily and immediately. Indeed, as teachers adapt the software to their particular educational circumstances and purposes, important variation is likely to arise in the ideas and methods associated with its use.

Through identifying mathematics departments which were professionally well regarded for their use of ICT in teaching, this study sought to examine relatively developed practice. However, despite official advocacy of the use of dynamic geometry in English secondary schools, uptake of the software proved to be quite recent in such departments. Typically, too, any school class was involved in at most a handful of lessons each year involving use of the software. Where such use was supported by written curriculum materials, these were locally produced. Consequently, the ways in which dynamic geometry was used in the teacher-nominated lessons that were studied would be categorised, under Laborde, 2001 typology, as remaining close to paper-and-pencil geometry.

As already noted, mathematics educators have tended to interpret geometry software as a means of supporting approaches to school mathematics based on relatively open student exploration, experimentation and investigation. However, pioneering projects have shown that teachers find it challenging to conceive appropriate tasks, and to manage student traversal of a mathematics curriculum through tackling them. Where dynamic geometry has entered mainstream classrooms, it appears to be used to support more established forms of pedagogical practice, notably student activity directed towards empirical confirmation of standard curricular results, often through guided discovery, as already prevalent in the teaching of geometry in many educational systems. An important interpretive flexibility is apparent, then, in the way in which conceptions of dynamic geometry differ between its pioneering advocates and more typical classroom practitioners.

Equally, interpretive flexibility is apparent in the variability in classroom practices that this study found to be associated with a widespread professional discourse of *employing dynamic geometry to support guided discovery*. First, the wider organisation of lessons was influenced by the considerations weighed and conclusions drawn by teachers in *evaluating the costs and benefits of student software use*. Essentially, the degree to which students themselves were expected to make use of dynamic geometry was influenced by the extent to which this was seen as *providing experience of a mathematical reference model* for the topic in question, and more fundamentally as *promoting mathematically disciplined interaction* with a generalised geometric system. Second, approaches to *handling apparent mathematical anomalies of software operation* were influenced by whether such anomalies were seen as providing opportunities to develop students' mathematical understanding, in line with a more fundamental pedagogical orientation towards *supporting learning through analysis of mathematical discrepancies.*

In effect, variability in students' exposure to the operation of dynamic geometry was driven by the degree to which developing their instrumental knowledge of the software was seen as promoting mathematical learning. This is particularly well illustrated by the contrast between the two cases which involved similar mathematical topics but featured distinctive classroom approaches, characterised in terms of *prepared teacher use of a dynamic figure to support persuasive mathematical presentation* [N] as against *structured student use of dynamic geometry to support mathematically principled interaction* [P1]. Where the development of new mathematical ideas was organised around teacher led presentation and questioning, with the teacher taking sole responsibility for use of the tool and exercising great care to avoid exposing apparent anomalies in its operation, the rationale was – in effect – to minimise instrumental demands on students; demands seen as unproductive, even obstructive, to mathematical learning. Where the development of new mathematical ideas was organised around student activity framed and guided by the teacher, with students' use of the tool structured in ways considered beneficial for building mathematical knowledge – including having students construct figures for themselves, and having them encounter and make mathematical sense of apparent anomalies of operation – the rationale was to optimise instrumental demands on students.

Such variation was related to differences in *positioning dynamic geometry in relation to curricular norms.* In those cases involving older classes closer to external examinations, the emphasis was on avoiding student work with the software or making it manageable, and on structuring dynamic geometry use more directly towards standard mathematical tasks. However, even in these latter cases, *incorporating dynamic manipulation into mathematical discourse* sometimes took classroom activity beyond established official norms. While dragging was often treated as a means of generating discrete examples of conventional static figures more efficiently than with manual tools, at other times it was used to focus attention on the dynamic variation of figural features and measures. In those cases – involving younger classes – where dynamic geometry served as a means of promoting mathematically disciplined interaction, teachers talked of the lesson following the national curriculum less tightly or going beyond it. The one marked breach of curricular norms related to *privileging a mathematical register for framing figural properties.* In the case in question [Q], the software served as a tool to operate in directly geometrical terms, rather than – as in the other cases – in the arithmetic terms where measurement provided the 'normal' means of framing and analysing figural properties.

In effect, in all but the outlier case [Q] in this study, commonalities in the interpretation of dynamic geometry were conditioned by established norms of English school mathematics: conceptualising the mathematical domain in question in terms of 'shape, space and measures', and employing an investigative method based on empirical induction from arithmetic patterns. Although earlier US work with the *Supposer* also employed this approach of deriving figural properties empirically through induction from measure patterns, it differed significantly – in line with the curricular culture of 'geometry' prevailing in the US – in treating such activity as a prelude to analysing properties in more directly geometrical and deductive terms; even if this was a transition which teachers found difficult to orchestrate. It was to this 'geometry' tradition, now largely dormant in English schools, that the outlier case [Q] in this study appealed. Indeed this case illustrates the significant learning required for students to make the transition to operating confidently in a geometrical rather than arithmetical register. More specifically, it displays an interpretation of dynamic geometry as a means of supporting such learning by capitalising on the way in which constructing figures with the software calls for explicit use of a geometrical register.

The analysis offered in this paper illustrates the interpretative flexibility surrounding the emergent use of dynamic geometry in English secondary schools. Differences in practice are conditioned by the varying degrees to which developing students' capacity to make use of this tool is valued for its contribution to their subject learning. This analysis suggests that explicit curricular recognition is required for the use of dynamic geometry as a disciplinary tool, and for the need for students to develop an associated instrumental knowledge, if dynamic geometry is to move from being a marginal amplifier of established practice to become a more integral organiser of a renewed practice of classroom mathematics.

Please cite this article in press as: Ruthven, K. et al, Constructions of dynamic geometry: A study ..., *Computers & Education* (2007), doi:10.1016/j.compedu.2007.05.013

Acknowledgements

We are especially grateful to the dedicated practitioners who made this study possible. Many thanks also to the UK Economic and Social Research Council for funding the project (R000239823), to Theresa Daly for her vital secretarial assistance, to Colette Laborde and Keith Jones for their comments on earlier working documents, and to the two anonymous reviewers acting for the journal. Earlier drafts of this paper were presented, after peer review, at the 2005 Computer Assisted Learning (CAL) conference, and at the 2006 British Educational Research Association (BERA) conference under the auspices of the Special Interest Group on New Technologies in Education.

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Please cite this article in press as: Ruthven, K. et al, Constructions of dynamic geometry: A study ..., *Computers & Education* (2007), doi:10.1016/j.compedu.2007.05.013