Chapter 6: Conceptualising mathematical knowledge in teaching

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Each of the preceding four chapters in this section of the book has examined the development of a particular line of thinking about mathematical knowledge in teaching. My task in this chapter is to offer a critical appreciation of these approaches, and to create a more overarching framework for synthesising their differing contributions to the analysis of key issues of policy and practice.

Subject knowledge differentiated

The first line of thinking can be described as *Subject knowledge differentiated* (Petrou & Goulding, Chapter 2). Its fundamental thrust is that expert teaching requires more than what would ordinarily constitute expert knowledge of a subject. Thus, its central concerns are to identify types of subject-related knowledge that are distinctive to teaching so as to develop a taxonomy of such knowledge. The goal is to provide an overarching heuristic framework that can guide the analysis, assessment and development of professional knowledge. This project has its roots in Shulman's pioneering sketch of a taxonomy of knowledge for teaching. Chapter 2 discusses subsequent work that has sought to refine Shulman's model in the light of more direct practical experience of assessing and developing mathematical knowledge in teaching.

Petrou and Goulding focus first on the way in which the range of subject-related aspects identified in the Shulman knowledge taxonomy was extended by the model of professional knowledge proposed by Fennema and Franke (1992). Because of the significance of *knowledge of learner cognitions* within the *Cognitively Guided Instruction* [CGI] approach with which Fennema and Franke had been involved, it is not surprising that they singled this out as a primary category of professional knowledge within their model. This of course reflected the much wider trend (as discussed more fully by Steinbring in Chapter 4) to conceive teaching less in terms of direct instruction and more in terms of indirect (radical constructivist) facilitation, or (social constructivist) mediation, of students' construction of knowledge. Likewise, sensitised to the crucial interaction between knowledge and beliefs through their experience of working with elementary-school teachers to develop the CGI instructional approach, Fennema and Franke incorporated teacher beliefs as well as teacher knowledge into their model. Finally, perhaps the most important feature of Fennema and Franke's model was their insistence on the need to acknowledge "the interactive and dynamic nature of teacher knowledge", and to examine it "as it occurs in the context of the classroom" (p. 162) (as discussed more fully by Hodgen in Chapter 3). Both the other (and later) programmes of research that Petrou and Goulding then go on to discuss in depth in Chapter 2 share this concern to attend to knowledge in classroom (inter)action; however, these programmes position themselves differently as regards the continuing centrality of the Shulman taxonomy.

From an extensive programme of research and development work conducted at the University of Michigan, Ball, Thames and Phelps (2008) have proposed a refinement of the Shulman taxonomy in which some of its core categories are further subdivided, and others reassigned. This refinement was informed by study of the way in which mathematical knowledge plays out in classroom practice, conducted with a view to developing operational measures of teacher knowledge. In the Michigan model, Shulman's *pedagogical content knowledge* [PCK] is conserved, but subdivided into *knowledge of content and students* [KCS], typically "an amalgam, involving a particular mathematical idea or procedure and familiarity with what students often think or do" (p. 401); and *knowledge of content and teaching* [KCT], typically "an amalgam, involving a particular mathematical idea or procedure and familiarity with pedagogical principles for teaching that particular content" (p. 402). Moreover, in this model, Shulman's *curricular knowledge* becomes a further subcategory of pedagogical content knowledge in the form of *knowledge of content and curriculum* [KCC]. Shulman's *content knowledge*, too, is conserved, as an overarching category of *subject matter knowledge* [SMK], but differentiated into *common content knowledge* [CCK] used in settings other than teaching (and so, of course, by non-teachers), and *specialised content knowledge* [SCK] which uniquely enables "teachers… to do a kind of mathematical work that others do not" (Ball et al., 2008, p. 400); for example, when they respond to students' questions and find a telling example to make a specific mathematical point; or when they modify tasks to make them either easier or harder by anticipating the effects of changing particular didactical variables that affect students' approaches and responses (which also implies a potential involvement of aspects of pedagogical content knowledge). Finally, the more tentative subcategory of *horizon content knowledge* [HCK] is proposed which concerns teacher awareness of how mathematical topics are related across the span of mathematics, and of how their development unfolds, as when teachers connect a topic being taught to topics from prior or future years, or explain how it will contribute to longer-term mathematical goals and purposes (although presumably such knowledge develops largely through encounters, as both student and teacher, with particular curriculum schemes and materials, raising the question of its relation to *knowledge of content and curriculum*).

Appraising this refinement of Shulman's taxonomy, Petrou and Goulding question the viability of demarcating specialised content knowledge from pedagogical content knowledge. Nevertheless, the taxonomic urge encourages fine distinctions:

> [S]izing up the nature of an error, especially an unfamiliar error, typically requires… flexible thinking about meaning in ways that are distinctive of specialized content knowledge. In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students. (Ball et al., 2008, p. 401).

Such examples also highlight the plurality of routes through which knowledge-in-use can emerge in teaching situations. Likewise, the use of 'amalgam' signals that many teaching problems cannot be adequately framed in 'pure' terms drawn from a single knowledge domain, or even by drawing on several domains independently. Put simply, satisfactory resolution of teaching problems must take account of, and often trade off between, interacting considerations of quite different types, framed in correspondingly different terms. This gives rise to solutions that often involve an irreducible fusion of such considerations, not reducible to the practice, or even logic, of any single pure knowledge domain. Moreover, for reasons both of ecological adaptation and cognitive economy, much professional knowledge comes to organise itself around paradigmatic problems and solutions that involve this type of fusion since these are closer to experienced teaching situations.

Thus, the contrasting approach taken at the University of Cambridge by Rowland, Huckstep and Thwaites (2003, 2005) has been to develop a taxonomy more directly grounded in analysis of teacher knowledge-in-use in the course of actual classroom teaching episodes. While the *Knowledge Quartet* acknowledges parallels to the Shulman knowledge taxonomy, it does not seek to refine that model. Rather, it is designed to provide a guide to mathematical knowledge-in-use that is well suited to supporting teachers' professional reflection and learning. This Cambridge taxonomy, like the Michigan one, establishes prototypical systems of classification rather than logical ones, evoked through paradigmatic examples more than formulated through tight definitions. Essentially, the *Knowledge Quartet* provides a repertoire of ideal types that provide a heuristic to guide attention to, and analysis of, mathematical knowledge-in-use within teaching. However, whereas the basic codes of the taxonomy are clearly grounded in prototypical teaching actions, their grouping to form a more discursive set of superordinate categories – Foundation, Transformation, Connection and Contingency – appears to risk introducing too great an interpretative flexibility unless these categories remain firmly anchored in grounded exemplars of the subordinate codes.

Petrou and Goulding conclude Chapter 2 by proposing a synthesis of the different taxonomies of teacher mathematical knowledge that they have reviewed. In my view, while attempting this task is a valuable exercise in comparing and clarifying the models, it is not one that can be completed satisfactorily. The comparability of categories from the different taxonomies is no less problematic than the distinction between categories within any one. Influenced by experience of transposing the *Knowledge Quartet* from an English to a Cypriot context, Petrou and Goulding's synthesis gives more priority to curriculum knowledge than does either the Michigan or the Cambridge model. In effect, Petrou and Goulding revert to something close to core elements of the original Shulman taxonomy, wherein curriculum knowledge sits alongside content (or subject matter) knowledge and pedagogical content knowledge. This provides a reasonable match to Ball et al.'s refinement of the Shulman taxonomy, but its fit to the *Knowledge Quartet* is more problematic.

The Shulman knowledge taxonomy and its subsequent variants have mesmerised the field rather at the expense of the model of pedagogical reasoning that accompanied early accounts of the taxonomy (Wilson, Shulman, & Richert, 1987). In particular, this model incorporates a crucial process of *transformation* which focuses on the *interpretation* and *representation* of disciplinary concepts, and on their *adaptation* to some general schooling situation and their *tailoring* to a particular group of students. If we view transformation as a process of problem solving, we see that it is subject to a range of constraints, both mathematical and pedagogical, which often cannot be considered in isolation from one another. Thus, what might appear to be simply a solution to a mathematical problem may have also been conditioned by pedagogical constraints and vice versa. Equally, where a teacher's solution to a problem of classroom teaching is also conditioned by curricular constraints, this can disguise a further interplay of mathematical and pedagogical considerations behind the institutionalised curriculum. Moreover, while solutions to such teaching problems may become crystallised as stable knowledge, they may equally be subject to continuing adaptation and refinement, and they will vary between teachers and across teaching settings. From this viewpoint, it becomes clearer why it has been so difficult to make demonstrable progress in establishing persuasive and productive knowledge taxonomies.

Petrou and Goulding's synthesis also incorporates the setting or context of teaching as an explicit, though relatively underdeveloped, element of their model. On the structural side, "the context in which teachers work is the structure that defines the components of knowledge central to mathematics teaching", and "this 'context' [includes] the educational system,… the curriculum and its associated materials, such as textbooks and the assessment system". On the agentic side, "teachers' [knowledge] can determine the ways in which [they] understand, interpret and use the mathematics curriculum and its associated materials". Petrou and Goulding also draw attention to the "largely individualistic assumption which underpins" the models of teacher knowledge that they have discussed and suggest that attention needs to be given to teacher knowledge in relation to the wider systems within which it functions and develops.

Subject knowledge contextualised

Analysing teacher (and teaching) knowledge from this wider perspective is the focus of the second line of thinking to be reviewed here; what can be described as *Subject knowledge contextualised* (Hodgen, Chapter 3). The fundamental thrust of this approach is that the (collective as well as individual) use and development of subject-related knowledge in teaching is strongly influenced by material and social context: the central concern of this approach is to identify and analyse significant facets of this contextual shaping. The goal is to acknowledge the embeddedness of knowledge in professional activity mediated by teaching tools and social organisation, so providing a model better adapted to guide the analysis, assessment and development of mathematical knowledge in teaching. This project as a whole draws on more general socio-cultural models, and Chapter 3 pursues this line of argument by using these models to characterise examples from studies of mathematical knowledge in teaching.

In Chapter 3, Hodgen seeks to illustrate the embeddedness of mathematical knowledge in professional practice, principally through the case of an experienced advisory teacher for primary mathematics who is actively involved in leading professional and resource development that includes intensive work on particular mathematical topics. Observed in her ordinary professional work, the teacher is able to function competently and confidently in tasks involving conceptual as well as computational mathematical activity. Observed in a more pressured interview situation involving apparently similar tasks, she does not display the same competence and confidence. While Hodgen notes that the teacher's capacity to access relevant knowledge may have been disrupted by anxiety triggered by the interview setting, the main explanation that he proposes is that the teacher's normal competence and confidence depends on the support ordinarily available to her through working collaboratively, using lesson materials, and drawing on curricular guidance.

Indeed, as Hodgen later points out, another important issue, already recognised by Shulman, is the form in which professional knowledge is held and the way in which it is organised and accessed. Here, cognitive studies of expert mathematics teaching at school level (Leinhardt et al., 1991) have found teachers' knowledge and reasoning about a particular topic to be organised in terms of 'curriculum scripts' closely tailored to the actual work of teaching; these memory structures provide loosely ordered repertoires of action and argumentation, including relevant representations and explanations as well as markers for anticipated student difficulties. Likewise, Hodgen suggests that restructuring existing knowledge and experience may play a more important part in learning to teach and developing as a teacher than acquiring wholly new knowledge. As noted earlier by Fennema and Franke, such restructuring is often likely to extend to belief as well as knowledge, so that teacher learning also involves a degree of reconstruction of identity.

While the approach developed in Chapter 3 offers a plausible broadening of perspective, it appears to rest as yet on a relatively slender and fragmentary evidential base. The other cases invoked are all treated much more briefly. One of them provides a counterpoint to the main case in showing the distribution of professional expertise across personal knowledge, teaching tools and social organisation. In it, the capacity of a high-school mathematics teacher to implement an innovatory reform-oriented curriculum appears to rest not just on his rich and well-articulated mathematical knowledge, exercised and perhaps further developed through teaching a traditional curriculum for many years, but also on the support provided by the curriculum materials and the professional development associated with the innovation. The other cases represent a particular type of dysfunction where the personal knowledge of teachers is not well-adapted to the particular context of school teaching. In the first of these, proficiency in academic mathematical practice does not provide prospective secondary teachers with ready resources to respond to a naïve question about what, for them, are taken-for-granted techniques, a type of question often posed to teachers in school mathematical practice. In the second of these cases, compacted knowledge of a mathematical topic accompanied by automated recognition of mathematical connections appear to impair the sensitivity of a middle-school mathematics teacher to his students' thinking, preventing him from connecting with, and making sense of, this thinking. In effect, these cases show not only that knowledge of more advanced mathematics does not, of itself, help a teacher to function effectively (for the reasons reviewed by Petrou and Goulding in Chapter 2), but that more elementary knowledge may have taken on a curtailed and automated character that stands in the way of effective functioning in many teaching situations. It seems that shifts in the preferred modalities of mathematical thinking associated with more advanced study can create 'expert blind spots' for teachers (Nathan & Petrosino, 2003; van Dooren, Verschaffel, & Onghena, 2002).

The approach to thinking about mathematical knowledge in teaching developed in Chapter 2 focuses on different facets of individual knowledge and understanding of mathematics which enable teachers to deploy the ideas and methods of the subject flexibly across teaching situations without reliance on contextual supports. While the approach developed in Chapter 3 accepts the desirability of teachers having these types of individual knowledge and understanding of mathematics, it acknowledges the reality that many school systems are obliged to operate with teachers who lack independent personal competence and confidence in the subject. Consequently, from a broader perspective which examines teaching in context, this approach suggests that teaching tools and social organisation provide potentially important mechanisms to help the teaching force function more effectively, and that these are also potentially capable of contributing to the development of teachers' individual knowledge and understanding of the mathematics that they are teaching.

Subject knowledge interactivated

The third line of thinking in this section on conceptualising mathematical knowledge in teaching can be described as *Subject knowledge interactivated* (Steinbring, Chapter 4). Unlike the other chapters, this one does not specifically examine teacher knowledge and learning, but focuses rather on an evolution of thinking about the character of mathematical knowledge and how it is mediated through teaching and learning. The fundamental thrust of the evolution that Steinbring describes is towards a view in which mathematical knowledge is taken to be only indirectly communicable and locally constructible through social interaction. Hence, the central concern of this approach is with the epistemic and interactional processes through which mathematical knowledge is (re)contextualised and (re)constructed in the classroom.

For Steinbring, this forms part of a more general view that emphasises reciprocal interaction between relatively autonomous systems, rather than direct action of one on another; a view applicable not just to relations between teacher and student, but also to those between teacher educator and classroom teacher, educational researcher and teacher developer. Nevertheless, by examining *Stoffdidaktik,* Steinbring is focusing on a tradition which plays a central part in the subject-related components of teacher education in the German-speaking world (Keitel, 1992). Thus, I will add to my précis of Steinbring's argument some observations of my own, based on my understanding of the form and function of *Stoffdidaktik* in teacher education, as gleaned from working with colleagues in German-speaking countries and from examining mathematics texts used in teacher education there.

Steinbring employs the widely-recognised 'didactical triangle', consisting of Mathematics/Content, Teacher/Teaching and Student/Learning (and the relations between them), as a device to compare three stages in the evolution of this central component of the German tradition of *Mathematikdidaktik*. In order to trace the evolutionary process, he describes changes at each stage in terms of the way in which these elements, and the relations between them, are conceived. At the first stage, that of classical *Stoffdidaktik*, attention was focused on Mathematics/Content, in particular on mathematically systematic analysis of curricular content to find an optimal presentation and sequencing for teaching purposes. Fundamental assumptions, then, were that achieving such a presentation is largely a problem of mathematical analysis, and that such a presentation then provides an unproblematic basis for effective teaching, and so for effective learning. Consequently, the other elements of the triangle, Teacher/Teaching and Student/Learning, received little attention at this stage, and the relationship between the elements was assumed to be a linear one in which well-analysed Content is relayed by the Teacher to the Student. One can see how, within teacher education, this view gave classical *Stoffdidaktik* a central place: a good teacher must be well-versed in the analysis of subject content for teaching purposes.

The second stage was precipitated by new views of the Student as a sense-making agent in the classroom, and of Learning as a process of knowledge construction. This was linked, in turn, to a new view of Mathematics/Content in which the processes and products of academic mathematical practice were less unquestioningly accorded a privileged place. Under these circumstances, the focus of attention within *Mathematikdidaktik* enlarged to include Student/Learning as well as Mathematics/Content, and the style in which Mathematics/Content was treated shifted (as exemplified by Freudenthal's influential work on didactical phenomenology and progressive mathematisation). These new views rather neglected the element of Teacher/Teaching, indeed sometimes treated it with a degree of suspicion. Within the institution of teacher education, reformed *Stoffdidaktik* responded by taking a broader perspective on Mathematics/Content and incorporating greater attention to Student/Learning, often in the form of the results of psychological studies of student errors and misconceptions (e.g., Padberg, 1989; Vollrath, 1994). In the third stage of the evolution described by Steinbring, attention has turned to direct analysis of knowledge construction in the classroom, particularly to its epistemic and interactional aspects. Although Mathematics/Content, Teacher/Teaching and Student/Learning are regarded as relatively independent systems, attention now focuses on their operation and particularly their reciprocal interaction. The type of analysis that results from this approach is illustrated in the final section of Chapter 4. Steinbring argues, of course, that such analyses carry no direct implications for teacher education or for teaching. Nevertheless, one can see how exposure to such examples might stimulate teacher educators to incorporate this type of fine-grained analysis of the construction of mathematical knowledge through classroom interaction into their work with prospective and serving teachers.

At one point, Steinbring introduces a comparison with the Anglo-American tradition. He suggests that there is a further type of mathematical knowledge relevant to teaching that goes beyond content knowledge and pedagogical content knowledge as characterised by Shulman:

what he terms *epistemological knowledge for mathematics teachers*. We can perhaps best understand this by first observing how Steinbring's use of the epistemological triangle draws attention to what otherwise might be taken as unproblematic mathematical entities. This approach highlights the way in which such entities are constituted by virtue of relationships between Concept, Sign/Symbol and Object/Reference Context, and how these relationships provide a key focus for the unfolding construction and negotiation of knowledge in classroom interaction. In effect, the epistemological triangle might be seen as a necessary expansion of the Mathematics/Content element within the didactical triangle to enable it to preserve its heuristic function given this new view which emphasises "the theoretical and dynamic character of mathematics". It is necessary, then, to treat with some caution apparent parallels between relations implied by the didactical triangle and the subcategories of pedagogical content knowledge in the refined (Michigan) version of the Shulman knowledge taxonomy. Superficially at least, Knowledge of Content and Students focuses on the relation (or, in this view, the interaction) between Mathematics/Content and Students/Learning; Knowledge of Content and Teaching focuses on the relation (or interaction) between Mathematics/Content and Teacher/Teaching. Knowledge of Content and Curriculum presumably resides within the Mathematics/Content element within the didactical triangle, but parallels are problematic given the way that the analysis presented by Steinbring has shown how views of this element have changed markedly within *Mathematikdidaktik*, as previously unacknowledged complexities have been recognised.

Subject knowledge mathematised

The final line of thinking developed in this section of the book can be described as *Subject knowledge mathematised* (Watson & Barton, Chapter 5). Its fundamental thrust is that teachers must act mathematically in order to enact mathematics with their students, and that doing so calls for a kind of knowledge rather different from that which normally receives emphasis in discussions of mathematical knowledge in teaching. The central concern of the chapter, then, is to characterise those *mathematical modes of enquiry* which underpin any authentic form of mathematical activity, and to show how teachers employ them to foster such activity in their classrooms.

While also embracing Krutetskii's 'mathematical abilities', the chapter locates its approach principally within a tradition leading from Polya's 'mathematical heuristic' to Cuoco, Goldenberg and Mark's 'mathematical habits of mind'. The chapter draws on each of these sources to exemplify the range of intellectual dispositions and strategies which can be thought of as mathematical modes of enquiry. The chapter then uses a simulated exercise in the planning of teaching situations from contrasting types of resource to provide more fully-developed examples of mathematical modes of enquiry in action within the work of teaching. Retrospective analysis of the thinking stimulated by this exercise also leads to more mathematical modes being identified. In an authentic piece of planning, of course, contextual aspects of the teaching situation would also figure, and might indeed shape key aspects of the process. Nevertheless, the artificiality of the exercise does help to focus attention on the mathematical modes of enquiry and their significance. In effect, what Watson and Barton are doing is "organizing the reality with mathematical means" (Freudenthal, 1973, p. 44), bringing out how both the enaction of teaching and its planning can be treated as processes of mathematising. This underpins their broader critique of predominant perspectives on mathematical knowledge in teaching, challenging their apparent focus on frozen mathematical content at the expense of fluid mathematical process (so taking further some of the critiques of Shulman's analysis reviewed in Chapter 2). Where teacher knowledge is concerned, the crux for Watson and Barton is that "a teacher for whom these [modes] are ways of being mathematical is more likely to be able to act fluently in all classroom mathematical contexts, compared with one who has learned a repertoire of pedagogical strategies without personal mathematical involvement".

Freudenthal (1991, p. 30) has described mathematisation as "the process by which reality is trimmed to the mathematican's needs and preferences" and this seems a very apt description of the approach set out in Chapter 5. A recurring feature of the argument is an emphasis on personal mathematical experience as a source of insight. Reviewing the planning exercise, Watson and Barton conclude that "the dominant knowledge brought to bear on the pedagogic tasks of planning and teaching was the personal mathematical past experience of the protagonists". But potential limitations of the personal mathematical experience and thinking of teachers as a guide to the mathematical experience and thinking of their students are not explored. Challenging the value of teachers learning about common mathematical misconceptions amongst students, Watson and Barton suggest that "personal experience of how misconceptions come to be constructed is a more powerful source of pedagogic knowledge". Arguing that mathematics teachers need to be able to "see behind" students' productions, they propose that this is exactly "what a mathematician might do when interpreting a text, or when thinking through a problem".

This approach comes close to a mode of thought that has been found to be prevalent amongst subject-specialist teachers, in which an explicit mathematical narrative provides the organising structure for a tacit pedagogical one.

> In mathematics teachers, the subject-matter-specific pedagogical content knowledge is to a large part tied to mathematical problems. In a way, it is "crystallized" in these problems, as research in everyday lesson planning has shown. In their lesson preparation, experienced mathematics teachers concentrate widely on the selection and sequence of mathematics problems… Nevertheless, pedagogical questions of shaping the lessons are also considered by teachers in their lesson planning, as these questions codetermine the decision about tasks. By choosing tasks with regard to their difficulty, their value for motivating students, or to illustrate difficult facts, and so forth, the logic of the subject matter is linked to teachers' assumptions about the logic of how the lesson will run, and how the students will learn.… Teachers often do not even realize the integration they effect by linking subject-matter knowledge to pedagogical knowledge. One example of this is their (factually incorrect) assumption that the subject matter (mathematics) already determines the sequence, the order, and the emphasis given to teaching topics. The pedagogical knowledge that flows in remains, in a way, unobserved. To teachers who see themselves more as mathematicians than as pedagogues, their teaching decisions appear to be founded "in the subject matter" (Bromme, 1994, p. 76).

Too mathematically purist a stance risks isolating discussion of mathematical knowledge in teaching from productive perspectives that are very much in sympathy with the idea of mathematical modes of enquiry, but which frame such ideas in different terms. For example, one of the cases which Collins, Brown and Newman (1989) use to illustrate their model of 'cognitive apprenticeship' is Schoenfeld's (1985) approach to teaching mathematical problem solving which lies in the same tradition of critical refinement of Polya's mathematical heuristic as does the mathematical modes of enquiry approach. Schoenfeld, however, is not uncomfortable with the language and concepts of the cognitive sciences; indeed, he uses that apparatus to generate fresh insights into teaching for mathematical problem solving. Others characterise his teaching in similar terms:

> To students, learning mathematics had meant learning a set of operational methods, what

> Schoenfeld calls *resources*. Schoenfeld's method [involves] teaching students that doing mathematics consists not only in applying problem-solving procedures but in reasoning about and managing problems using heuristics, control strategies and beliefs. Schoenfeld's teaching employs the elements of modelling, coaching, scaffolding and fading in a variety of activities designed to highlight different aspects of the cognitive processes and knowledge structures required for expertise (Collins et al., 1989, p. 470).

The language employed by Watson and Barton in the Conclusion to Chapter 5 suggests that theirs is not a purist stance. For example, the references to "authentic mathematical experiences", "having modes modelled", and "having the modes explicitly discussed at a meta-level" appear to borrow from the same forms of language as cognitive apprenticeship. Equally, Collins et al. (1989, p. 474) acknowledge how mathematical and pedagogical considerations (of exactly the type identified by Bromme in the quotation above) interact in Schoenfeld's thinking about his teaching: "Schoenfeld places a unique emphasis on the careful sequencing of problems. He has designed problem sequences to achieve four pedagogical goals: motivation, exemplification, practice, and integration."

Reconceptualising subject knowledge in teaching

Originating in response to perceived inadequacies in received views of mathematical knowledge in teaching, each of the lines of thinking presented in Chapters 2 to 5 has given rise to productive reconceptualisations.

The first pair of approaches focus on how mathematical knowledge can be functionally adapted and developed in ways that specifically support the teaching role. *Subject knowledge differentiated* challenges the received idea that the mathematical knowledge required in teaching is simply that developed through studying the subject to a level which provides adequate facility in (and perspective on) the material to be taught. This approach has reconceived the issue productively by identifying types of subject knowledge that are closely linked to teacher activity in promoting effective learning, which are not normally developed as a student of the subject, and that appear to be distinctive to teaching (or, at least, developed much more fully than in other professions). This also introduces a challenge to the received idea that teaching simply involves the direct application of mathematical knowledge that is universal in its character and organisation. *Subject knowledge contextualised* further reconceives this issue by showing how teachers' knowledge undergoes a professionally-specific adaptation, in which organising structures are developed that support coordinated attention to the mathematical and pedagogical facets of teaching, and that involve an important degree of (sometimes unacknowledged) fusion between mathematical and pedagogical concepts. *Subject knowledge contextualised* has also challenged what appears to be an unexamined emphasis in received views of mathematical knowledge in teaching on such knowledge as individually and independently held. In the face of widespread difficulties in securing and developing a mathematically proficient workforce within teaching, this issue has been reconceived productively by drawing attention to the distribution of subject knowledge across teaching tools and professional communities, and so to the contribution that (when appropriately developed and organised) these resources can make to supporting knowledgeable subject teaching and the development of teachers' subject knowledge.

The second pair of approaches share an emphasis on a teaching role that centres on supporting student knowledge construction. *Subject knowledge interactivated* challenges the received idea that mathematical knowledge can be preformulated in a way that enables it to be simply relayed by teachers to students (or indeed by teacher educators to prospective teachers). This approach has reconceived the issue productively by showing the types of interactional process through which mathematical knowledge undergoes a process of (re)construction by means of active negotiation between the participants in classroom mathematical activity, mediated by appropriately designed tasks. In particular, this highlights the types of subject-specific epistemic and interactional competence that are required in effective teaching of this style. Closely related is the challenge which *Subject knowledge mathematised* offers to the received idea that it is sufficient for teachers to know their subject in the sense simply of being familiar with the finished mathematical material to be taught and with associated difficulties that students may encounter. This approach reconceives the issue productively by highlighting the way in which teachers are responsible for leading classroom enactment of mathematical activity through which knowledge can be (re)constructed, and by illustrating how, at its best, such joint activity provides a means through which students become conversant with the mathematical modes of enquiry that underpin such (re)construction (i.e., how they can be led to develop syntactic as well as substantive competence (Schwab, 1978)).

In the face of the widespread difficulties noted earlier in securing and developing a mathematically proficient workforce within teaching, this latter pair of approaches might be seen as setting a somewhat utopian (and overly challenging) standard for mathematical knowledge in teaching. The counter to this is that more modest strategies may only be capable of effecting marginal improvement within received practices of mathematics teaching which are fundamentally flawed. Indeed, from the more radical perspective, the problem of subject expertise in teaching is just one component of a much larger issue of the social reproduction of mathematical knowledge. In this view, inadequate mathematical knowledge on the part of individual teachers is a subsidiary phenomenon that ultimately resides in the inadequacies of received practices, not just of mathematics teaching but of mathematical communication more broadly, because these lack mechanisms through which the thinking processes and learning strategies that underpin the development of mathematical knowledge are made accessible to students and significant to their teachers.

Within the practice of academic mathematics, for example, Thurston (1994, p. 8) has argued that established protocols for communication fail to provide effective means of revealing underlying thinking processes:

> We mathematicians need to put far greater effort into communicating mathematical *ideas*. To accomplish this, we need to pay much more attention to communicating not just our definitions, theorems, and proofs, but also our ways of thinking. We need to appreciate the value of different ways of thinking about the same mathematical structure. We need to focus far more energy on understanding and explaining the basic mental infrastructure of mathematics… This entails developing mathematical language that is effective for the radical purpose of conveying ideas to people who don't already know them.

Thurston is arguing that effective mathematical communication involves some degree of interactivation and mathematisation of the knowledge at stake. In particular, he criticises approaches to teaching mathematics that neglect such interactivation in favour of a reductive focus on demathematised knowledge.

> In classrooms… we go through the motions of saying for the record what we think the students "ought" to learn, while the students are trying to grapple with the more fundamental issues of learning our language and guessing at our mental models. Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material "covered" in the course, and then grading the homework and tests on a scale that requires little understanding. We assume that the problem is with the students rather than with communication: that the students either just don't have what it

takes, or else just don't care. Outsiders are amazed at this phenomenon, but within the mathematical community, we dismiss it with shrugs (p. 6).

This highlights how (presumably excellent) conventional mathematical knowledge on the part of (university) teachers is unable to compensate for what Thurston characterises as ‘often dysfunctional’ cultural practice. Translated into the language of cognitive apprenticeship, the type of teaching practice sketched above models mathematical activity in restricted terms, employs excessive scaffolding without progressive fading, and neglects cognitive and metacognitive articulation and reflection. The result is impoverished joint activity, weak interaction between teacher and students, and a corresponding polarisation of their classroom roles that accentuates the shortfall of (particularly tacit) knowledge to which students are given access.

Under such conditions, few students will develop powerful and flexible strategies of mathematical thinking and learning. A fundamental problem of mathematical knowledge in teaching is that the school and university experience of many prospective and practising teachers has been of this limited type, creating reflexes that are difficult to change. The models of subject thinking and learning that prospective teachers have developed as students are well known to constitute an important base for the forms of teaching practice that they go on to develop. Arguably indeed, taking on a teaching role involves a recasting of intrapersonal metacognition into interpersonal activity and dialogue. Interestingly too, this argument suggests that the degree to which (what have been presumed to be) distinctive elements of mathematical knowledge for teaching can be differentiated from other mathematical knowledge may actually be mediated by cultural practices of teaching. Specifically, the degree of differentiation between what is regarded as teaching-specific, rather than as more generic, mathematical expertise may be directly related to the asymmetry of teacher and student roles in classroom mathematical activity, and inversely related to the attention accorded there to cognitive and metacognitive articulation and reflection. That provides one of the reasons why we have been keen in this book, particularly in the next section, to extend our consideration of issues of mathematical knowledge in teaching beyond the English-speaking world in which the predominant conceptualisations reviewed in Chapters 2 and 3 have been developed.

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