

Linking algebraic and geometric reasoning with dynamic geometry software: Phase 2 - classroom work

Report to the Qualifications and Curriculum Authority

Kenneth Ruthven (University of Cambridge),

Adrian Oldknow (Chichester University College) &

Ron Taylor (Hampshire Inspection and Advisory Service), with

Carol Knights (Applemore School),

Paul Barber, Jill Barton & Clare Dolan (Cams Hill School),

Peter Ransom (Mountbatten School),

Graeme Bradshaw & Dawn King (Neville Lovett School)

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0 Contents

1	Introduction to Phase 2 of the study
2	<i>Dynamic Number Line</i>
2.1	<i>Dynamic Number Line</i> in Phase 1
2.2	<i>Dynamic Number Line</i> at Cams Hill
2.2	<i>Dynamic Number Line</i> at Applemore
3	<i>Bounded Circle</i>
3.1	<i>Bounded Circle</i> in Phase 1
3.2	<i>Bounded Circle</i> at Neville Lovett
4	<i>Folded Paper</i>
4.1	<i>Folded Paper</i> in Phase 1
4.2	<i>Folded Paper</i> at Mountbatten
5	<i>Midpoint Locus</i>
5.1	<i>Midpoint Locus</i> in Phase 1
5.2	<i>Midpoint Locus</i> at Cams Hill
6	Other activities
6.1	<i>Pythagoras Rule</i> at Neville Lovett
6.2	<i>The Baywatch Problem</i> at Applemore
6.3	<i>Curve Gradients</i> at Applemore
6.4	<i>Coordinate Designs</i> at Cams Hill
7	Overview and conclusions
8	Acknowledgments

1 Introduction to the Phase 2 of the study

This study has investigated the potential of dynamic geometry software (DGS) to support curricular activities which link algebraic and geometric reasoning. It forms one element of a QCA project examining how the teaching of algebra and geometry can be strengthened within the framework of the National Curriculum for Mathematics.

The report on Phase 1 of the study offered an *a priori* analysis of the potential of DGS to support algebraic and geometric reasoning, in the light of new developments in the capabilities of such software. It included specific ideas for classroom mathematical activities, devised to illustrate possibilities of using DGS to link algebraic and geometric reasoning, and intended to provide the basis for future work with teachers.

Phase 2 of the study –the subject of this report– initiated work to develop the Phase 1 material into viable and valuable classroom teaching and learning activities. QCA funding supported the involvement in this work of the three project leaders and the seven participating teachers –from four Hampshire schools– over a period of eight weeks during the autumn term of 2003. The group held after-school meetings on 14 October and 25 November. At the first of these meetings, ideas and activities from the Phase 1 report were reviewed and discussed; at the second, classroom work undertaken by the teachers was reported and discussed. Subsequently, the participating teachers submitted written reports on their work, and these have provided the basis for the lesson descriptions provided in this report.

The table below summarises the lessons or lesson segments developed and trialled by teachers. In the sections which follow, this material has been organised according to activity, and –where appropriate– reference has been made to relevant material from the Phase 1 report.

School	Teacher(s)	Activity	Class(es)	Technology use(s)	Duration
Applemore	Carol Knights	<i>Dynamic Number Line</i>	Y9 top set Y10 top set	Classroom projection	Lesson starter
Applemore	Carol Knights	<i>Baywatch Problem</i>	Y9 top set	Classroom projection Independent pupil use	Unit of 4 lessons
Applemore	Carol Knights	<i>Curve Gradients</i>	Y10 top set	Classroom projection Independent pupil use	Unit of 4 lessons
Cams Hill	Paul Barber, Jill Barton, Clare Dolan	<i>Dynamic Number Line</i>	Y9 & Y10 varied sets	Classroom projection	Lesson starter
Cams Hill	Paul Barber	<i>Midpoint Locus</i>	Y9 mid set Y10 top set	Classroom projection	Whole lesson
Cams Hill	Paul Barber	<i>Coordinate Designs</i>	Y7 low set Y8 low set	Classroom projection Independent pupil use	Whole lesson
Mountbatten	Peter Ransom	<i>Folded Paper</i>	Y9 low set Y10 top set	Classroom projection	Lesson launch
Neville Lovett	Graeme Bradshaw	<i>Pythagoras Rule</i>	Y9 set 2	Classroom projection	Lesson launch
Neville Lovett	Dawn King	<i>Bounded Circle</i>	Y9 top set	Classroom projection	Lesson plenary

2 Dynamic Number Line

2.1 Dynamic Number Line in Phase 1

In the Phase 1 report, it was suggested that the algebraic idea of letter as variable be represented through construction of a dynamic number line. This is a number line on which an independent variable and its dependent variables are plotted; as the point representing the independent variable is dragged along the line, the points representing the dependent variables move accordingly.

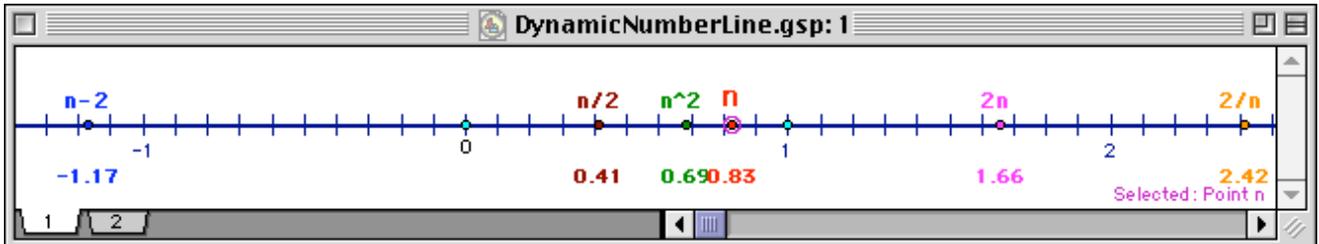


Figure 2.1.1: Screensnap of a dynamic number line showing an independent variable n and dependent variables $n-2$, $2n$, n^2 , $n/2$ and $2/n$.

In the example shown in Figure 2.1.1, the independent variable is n , and the dependent variables are $n-2$, $2n$, n^2 , $n/2$ and $2/n$ respectively. (For those reading this in monochrome, the points are colour-coded as well as labelled on the computer screen, to assist discrimination between them.) As presented here, the value of a variable is shown in two ways –as a position on the number line, and as an associated decimal number. As the independent variable n is dragged along the number line, the dependent variables move correspondingly. From the starting position shown, as n increases – moves to the right- all other variables do likewise with the exception of $2/n$ which decreases – moves to the left. Likewise, as n decreases, so initially do all the other variables except $2/n$. However, as n passes through 0, the direction of movement of n^2 reverses; it seems to ‘bounce off’ the 0 position; in addition, n^2 ‘overtakes’ n as they pass together through 1. Similarly, as $n/2$, n , and $2n$ pass through 0, their relative positions reverse; and these three variables move along the number line at different, but clearly related, ‘speeds’. By contrast, $n-2$ never exchanges position with n ; rather, both move at identical ‘speed’, and maintain a fixed ‘distance’. Equally, important insights can be gained by focusing on instantaneous relationships between the values of variables; for example, n and $2/n$ coincide just as n^2 passes through 2 (as shown in Figure 2.1.2).

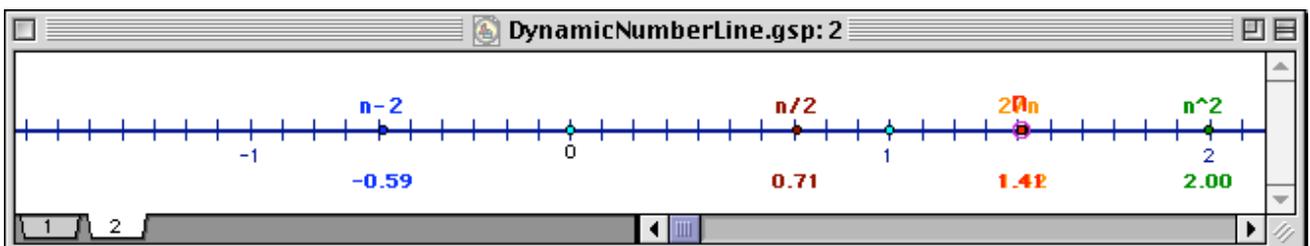


Figure 2.1.2: Screensnap of a dynamic number line showing an instantaneous relationship between the values of n , $2/n$ and n^2 .

This dynamic representation of changing variables can serve, then, to highlight important mathematical phenomena calling for investigation and explanation, and to activate important supporting metaphors. And the ideas involved are precisely those underpinning fundamental issues including the solution of inequations such as $n^2 > n$, the non-solvability of equations such as $n = n-2$, and the relative rates of change of independent and dependent variables.

2.2 Dynamic Number Line at Cams Hill

At Cams Hill, a *Dynamic Number Line* activity was used as a lesson starter with a wide range of classes in Years 9 and 10. Diagrams were displayed to the whole class by means of an interactive whiteboard or data projector. The original display had been simplified by placing only one dependent variable on each line. This was introduced to pupils as a way of showing an ‘input’ variable n and an ‘output’ variable dependent on a ‘rule’ in n : (See Figures 2.2.1 & 2.2.2).

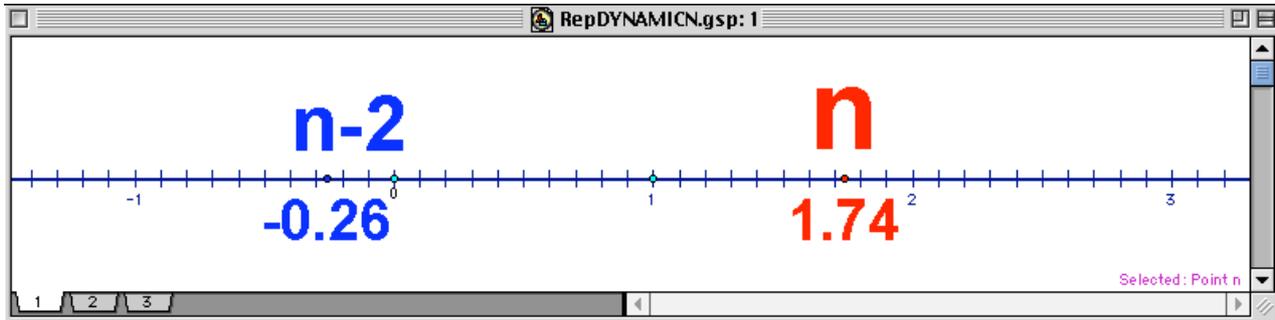


Figure 2.2.1: Screensnap of a dynamic number line displaying n and $n-2$.

Pupils were asked to ‘predict the output of the rule’ for particular values of n , beginning with positive integers and then moving on to negative integers and decimals. Results were checked by dragging n to the corresponding position on the number line. Following this work on specific ‘input-output’ pairs, pupils were asked to describe how the movement of the ‘input’ variable on the dynamic number line was related to the movement of the ‘output’ variable.

For positive integer inputs, and the easier rules $n-2$, $2n$ and $n/2$, pupils found it fairly simple to predict outputs. Predicting the outputs for decimal and negative inputs required more discussion. When a class moved on to tackle the more difficult rules, n^2 and $2/n$, pupils had to think more carefully even about positive inputs. Pupils found the task of describing movement more challenging and needed to discuss specific inputs and outputs when attempting to convince each other of their descriptions of movement.

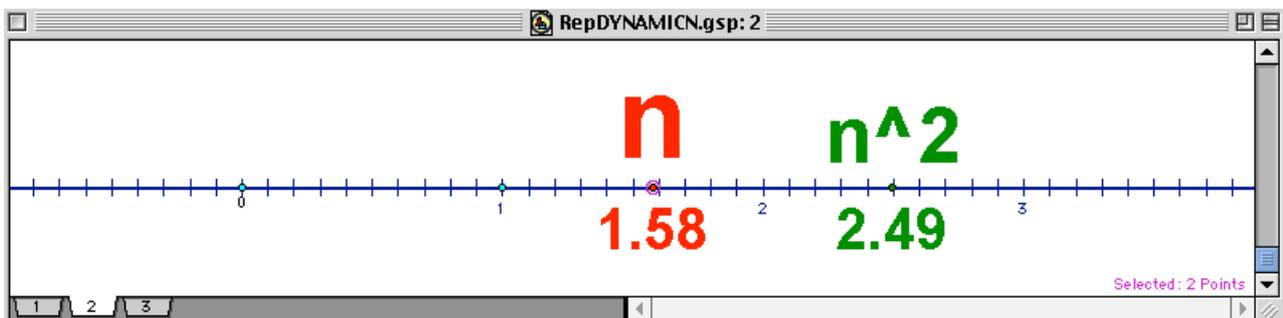


Figure 2.2.2: Screensnap of a dynamic number line displaying n and n^2 .

Reviewing the contribution which these publicly displayed dynamic number lines made to the lessons, teachers reported that “Pupils commented on the usefulness of being able to actually move the numbers and see the resulting movement of the connected rule allowing them to quickly confirm or discard their predictions”. Teachers considered that this approach “allows an excellent visual display of connections between rules and their inputs” and commented on the way in which it became possible to “consider the overall effect as opposed to individual cases or a small sample”. It also seems that use of the dynamic display helped to stimulate class discussion and to assist the formulation and testing of mathematical conjectures.

2.3 Dynamic Number Line at Applemore

At Applemore, the original activity was adapted for use with a Year 10 top set as a 10-minute lesson starter. Again, the line was projected from the teacher's laptop onto the classroom white board. However, a variant form was employed. On the dynamic number line, output variables were given arbitrary labels, so disguising the rule linking them to the input variable n . (See Figure 2.3.1).

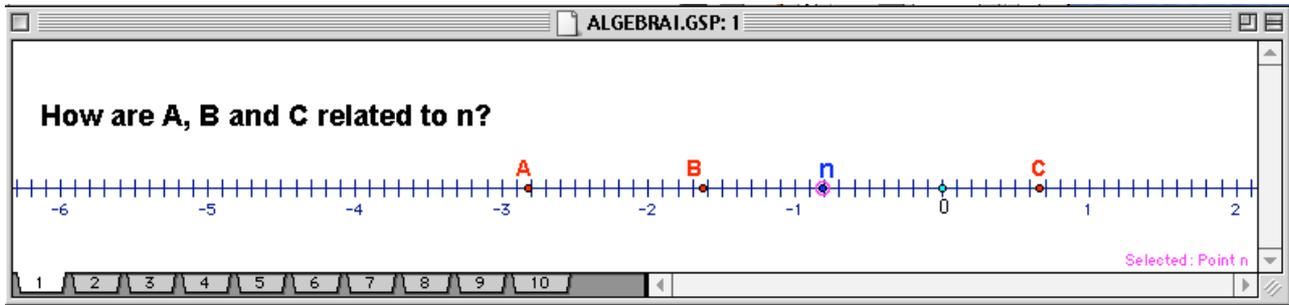


Figure 2.3.1: Screensnap of a dynamic number line displaying unspecified variables dependent on n .

The teacher saw the activity as a means of promoting algebraic thinking and reasoning. She explained to the class that n was a variable which she could move along the number line, and set the task of discovering (as a group) the relationship of A, B, C to n . Pupils suggested several alternative 'formulae' for A. They then asked for n to be moved to a certain value, explaining why they had chosen that particular value. When n was moved to that value they were able to eliminate certain formulae and arrive at the correct one. A similar process was used for B and C, and class discussion examined the merits of using zero as a value to ascertain the constant term, if there was one. Once it had been established that C was n^2 , the teacher questioned pupils further about the relationship between n and C. She asked if they could suggest values for n such that $C < n$. They then came up with the range of values to fit this criterion.

Reviewing the lesson, the teacher reported that an important benefit of using ICT for this activity was the dynamic quality of the display which enabled pupils to see that for A ($n-2$) the distance between A and n was fixed. She commented that "The class 'owned' the task – they were motivated because they controlled it. I could have done a similar task and used myself as a number machine but then I would have been the 'expert' with the answers." The teacher also valued having a visual representation of how variables moved in relation to each other, commenting on how pupils had noted the variation in the speed of C (n^2) as n was dragged steadily along the line. She even reported that "One student referred to the shape of the quadratic graph being a parabola and linked this to C's motion." Finally, the teacher noted that "It was very easy to verify the values for which $C < n$."

3 *Bounded Circle*

3.1 *Bounded Circle in Phase 1*

In the Phase 1 Report, *Bounded Circle* served to illustrate the basic idea of graphing covarying measures within a geometrical construction. In this case, a circle is constructed, and then two regular hexagons; an 'inner' hexagon with vertices on the circle, and an 'outer' hexagon with edges touching the circle (as shown in Figure 3.1.1). Variation—in the form of a simple scaling operation—is introduced by dragging the point defining the radius of the circle. The covariation of any two measures can be shown by plotting them as a coordinate pair. Dragging the radius in the geometrical figure results in the plotted point moving accordingly. This movement can be traced, showing how 'inner' perimeter, circumference and 'outer' perimeter covary with radius. The invariance of the visual nesting of 'inner' hexagon within circle within 'outer' hexagon is reflected in the numeric ordering of measures of perimeter/circumference. Even where pupils have already grasped these relationships, this simple situation can serve to illustrate the idea of covariation of measures.

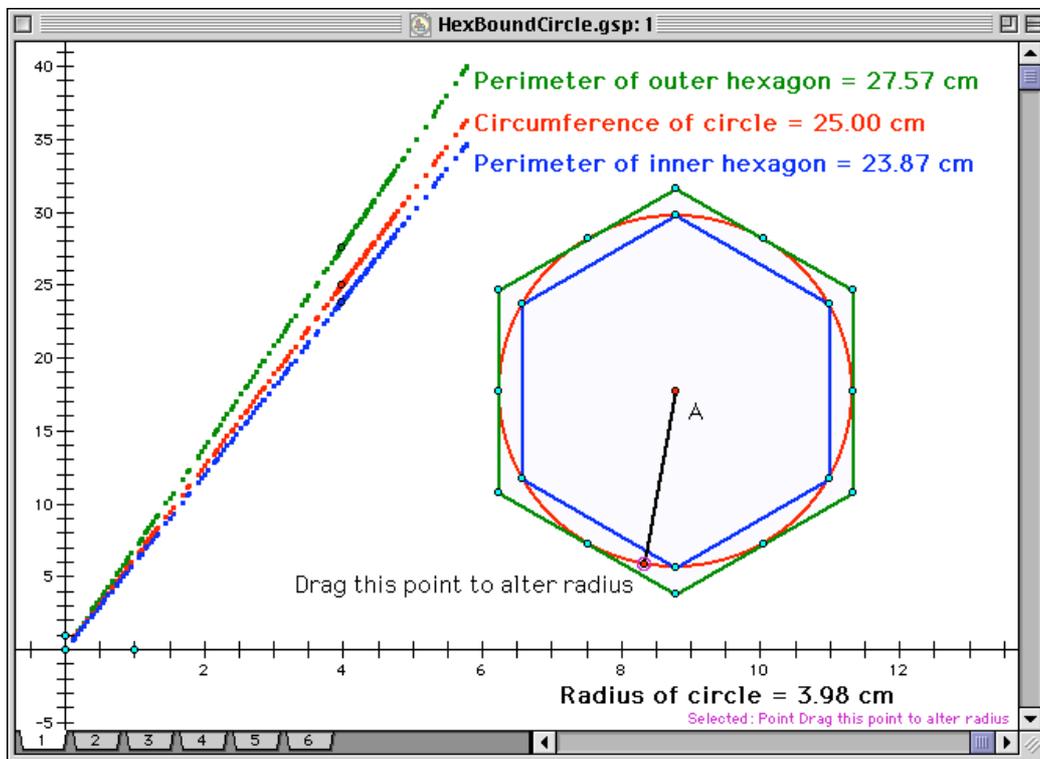


Figure 3.1.1: Screensnap showing covariation of perimeter/circumference with radius measures.

3.2 Bounded Circle at Neville Lovett

At Neville Lovett, a variant of *Bounded Circle* was used with a Year 9 top set. Earlier in the term this class had completed a pencil-and-paper activity where successive approximations for the circumference of a circle had been found by bounding it by an inner and outer regular polygon, and finding the mean perimeter of these polygons. Starting with two regular pentagons, the number of polygon sides has been incremented step by step, allowing closer and closer approximations to be made to the circumference of the circle. With this earlier work in mind, the teacher chose to use *Bounded Circle* with this class.

Taking account of pupils' limited knowledge of DGS, the teacher planned a lesson in which they would carry out the activity on paper, with use of the dynamic model being reserved for a plenary to draw ideas together at the end of the lesson. The aims of the lesson were to consolidate pupils' understanding of circumference; to introduce the idea of co-varying measures; and to establish the link between the geometric model and the graph of measures.

The pupils were given a worksheet with 5 circles of radii from 2cm to 6 cm bounded by two hexagons. They were asked to measure and record the perimeter of the two hexagons and calculate the circumference of the circle, using the formula $2\pi r$. (For the closing plenary, a DGS sheet was used to summarise these results. See Figure 3.2.1)

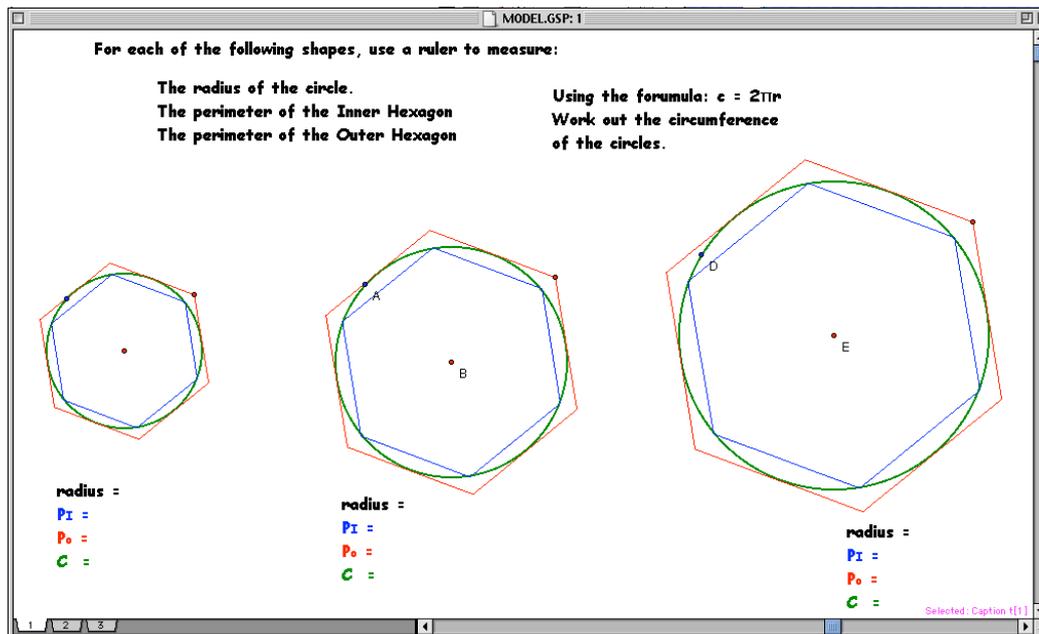


Figure 3.2.1: Screensnap of the template for recording worksheet results.

Pupils were asked to plot their results on a graph, using a colour code for each shape, so that they could compare the results and comment on them. From the comments made by pupils, it was clear that they could make a very visual link between the dynamic geometric diagram and the line graphs of measures. They commented on the fact that the straight line graph sandwiched between the two others corresponded to the circle sandwiched between the two hexagons. (A DGS sheet based on Figure 3.1.1 was used to illustrate this dynamically in the closing plenary).

Reviewing the lesson, the teacher considered that this activity has a lot of potential. She envisaged developing it further to examine the gradients of the graphs in order to establish the relationship between the radius of the circle and the circumference/perimeters of the circle/hexagons. With pupils more familiar with using a DGS, the dynamic diagram and graph could be constructed by the pupils, and they could look at what happens when the number of sides on the polygons are varied.

4 *Folded Paper*

4.1 *Folded Paper in Phase 1*

In Phase 1, *Folded Paper* was used to illustrate how a dynamic geometry figure can support various forms of geometric, algebraic and trigonometric reasoning. The basic construction shows the outline of a sheet of paper folded over so that one of its vertices touches the opposite edge, with the paper flattened to form a crease. In the basic construction (Figure 4.1.1), this vertex P can be dragged so that the endpoints of the crease remain within the boundaries defined by the edges AB and BC. The dimensions of the sheet of paper can also be altered by dragging C and D, introducing further generality to the situation.

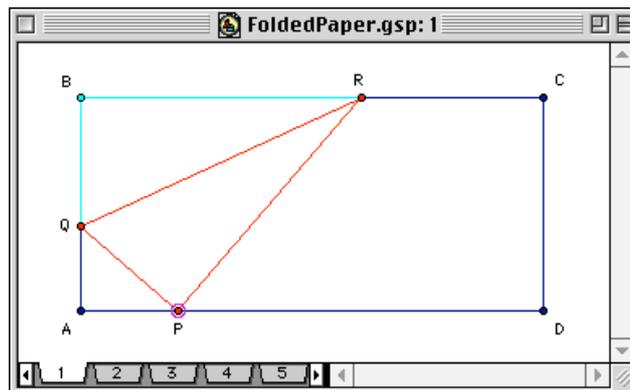


Figure 4.1.1: Screensnap of the basic folded paper construction.

One suggested task focused on angle relationships. With the measures of two angles displayed (as in Figure 4.1.2), the task is to identify and analyse any consistent relationship between them as P is dragged. This depends on using standard angle properties (notably those of complementary, supplementary and vertically opposite angles, and angles in a triangle) and can be developed into a systematic relationship expressed in algebraic terms. An extended task is then to conduct a complete analysis of angle relationships in the figure, so determining the minimum number of angle measures needed to deduce all others, and thus the minimum number of letters needed to express all angle measure variables. A similar task focuses on identifying systematic relationships between lengths within the figure and expressing them algebraically (with supporting mensuration as shown in Figure 4.1.3).

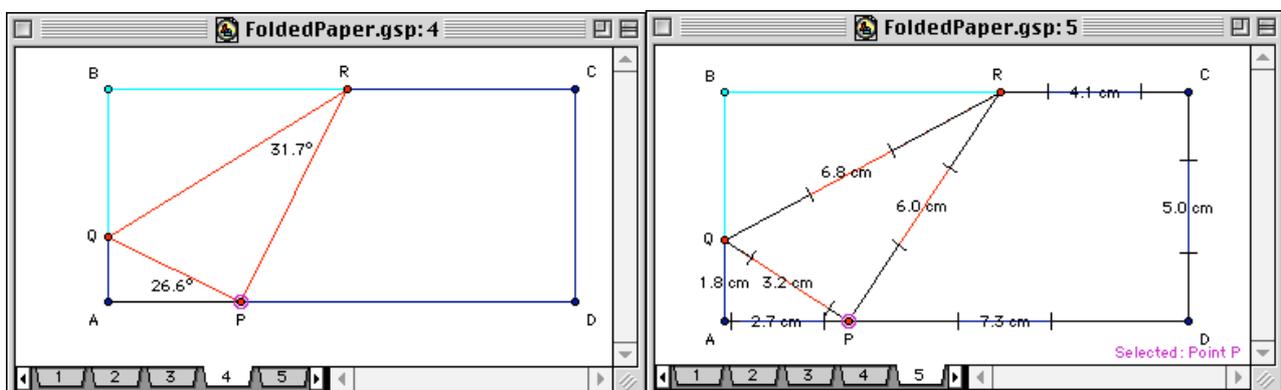


Figure 4.1.2-3: Screensnaps of basic displays for exploration and analysis of angle and length relationships .

4.2 *Folded Paper at Mountbatten*

At Mountbatten, the participating teacher was attracted to *Folded Paper* because he felt that it would be accessible to a wide range of attainers, and that the tactile activity of actually folding and manipulating paper would appeal to “kinaesthetic learners”. He used the basic idea in two successive lessons, adapting it to the differing characteristics of the classes concerned. The first class was a Year 10 set of high attainers: here, the teacher wanted them to do some ‘angle chasing’, leading into generalising by using algebra. The second class was a Year 9 set of low/middle attainers; here the teacher was primarily concerned with calculating angles, judging that trying to cope with the algebra would have got in the way of the geometrical reasoning for this particular class.

The teacher made use of the dynamic geometry figures at the start of the lesson in order to introduce the activity, providing demonstration and motivation to help pupils get started. He reported that it was useful to have accurate diagrams projected onto the whiteboard for pupils to see, and that the use of ICT made it easy to switch between diagrams and helped to focus pupils’ attention. The teacher then asked pupils to work in pairs, using real sheets of paper to tackle the activity and record their results. He noted that this enabled pupils to discuss their reasoning and gave him the opportunity to move around and help out. At the end of each lesson pupils were asked to ‘Write a sentence about what you think you have learnt today’ and to rate the lesson on a 0 (Sucks) to 10 (Rocks) scale.

In the opening segment of the Year 10 lesson, pupils’ attention was drawn to key features of the three GSP configurations illustrated above:

- (Figure 4.1.1): Move the point so they can see what happens. Ask what shapes they can see – will they always be that way? Are there any limitations? (This will revise their shape vocabulary.)
- (Figure 4.1.2): Where are there right angles? Mark them on their sheets. Can they fill in the other angles with algebraic expressions? See if they can prove which angles are equal.
- (Figure 4.1.3): Ask them to use properties and Pythagoras to give algebraic expressions for the sides.

The teacher was pleased with the way in which the pupils engaged with the activity and with the quality of the work that they produced. The distribution of pupil ratings of the lesson was positively skewed, ranging from 4 to 10, with lower quartile 5, median 8, and upper quartile 9. Typical pupil comments were “I learnt about angles and how it fits in with algebra” and “I have learnt how to use Pythagoras in lots of different ways. Good fun, made a change from working in books. Motivating!!!”.

Similarly, in the opening segment of the Year 9 lesson, pupils’ attention was drawn to key features of two of the GSP sheets illustrated above:

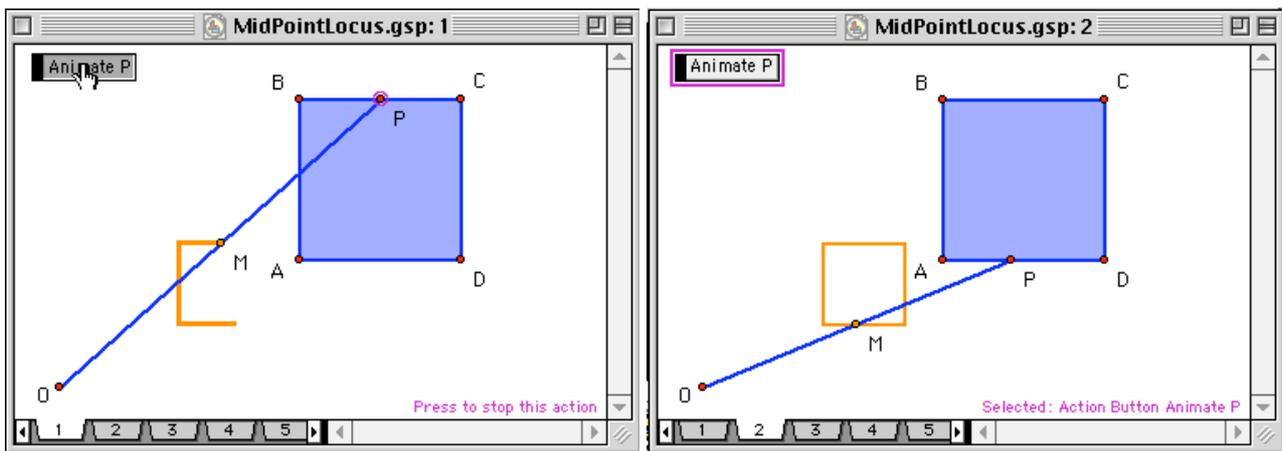
- (Figure 4.1.1): Move the point so they can see what happens. Ask what shapes they can see – will they always be that way? Are there any limitations? (This will revise their shape vocabulary.)
- (Figure 4.1.2): Where are there right angles? Mark them on their sheets. If I say this angle is 40° , can they work out the other angles? Check this, and then ask them to repeat with 52° .

The teacher felt that this lesson was less successful than the previous one. He reported having problems with two pupils who cause trouble regularly and noted that this may have had an influence. Nevertheless, most pupils produced analyses of two different foldings of the paper, and some annotated their work to indicate the mathematical principles behind their calculations. The distribution of pupil ratings was wide, ranging from 0 to 9, with lower quartile 2, median 5, upper quartile 8. Typical comments were “I have learnt how to calculate angles” and “Learnt today [to] cacluate [sic] all angles in triangles and quadrilaterals”.

5 Midpoint Locus

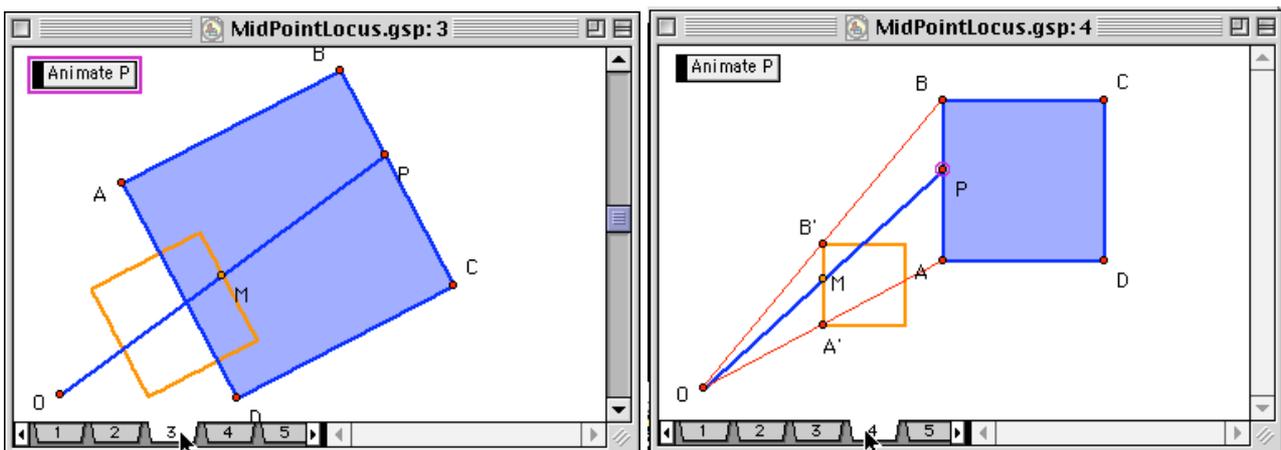
5.1 Midpoint Locus in Phase 1

In Phase 1 Report, *Midpoint Locus* was used to illustrate opportunities for parallel analyses of locus problems in classical, transformational and co-ordinate terms –with the last linked to algebraic methods. The problem situation concerns the locus formed by midpoint of a line segment connecting a fixed point (in effect, an origin) to a point moving round the perimeter of a square.



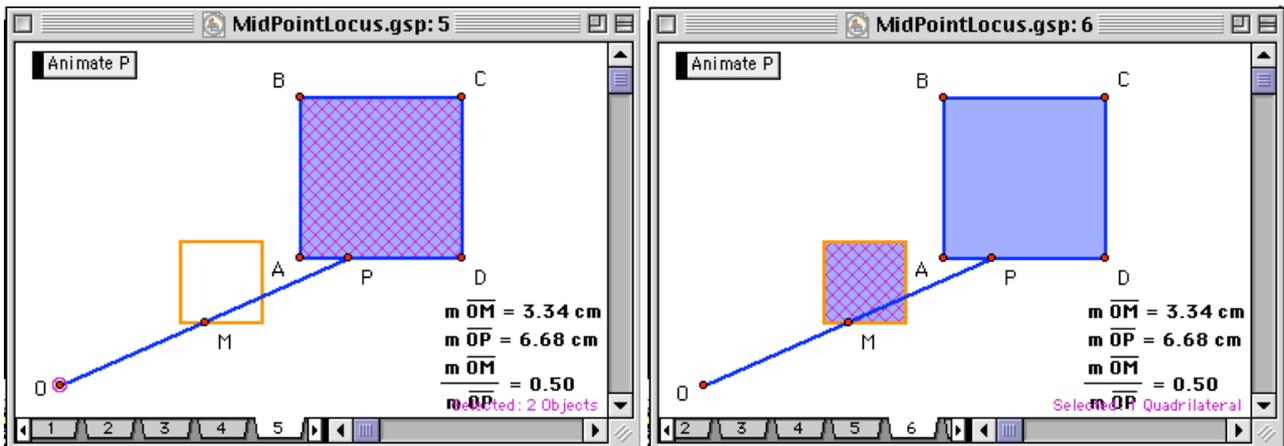
Figures 5.1.1-2: Locus formed by the midpoint, M , of a line segment connecting a fixed point, O , to a point, P , moving round the perimeter of a square.

As the point P moves (as animated in Figures 5.1.1-2), so does the midpoint M (as shown by its trace) in what appears to be a smaller version of the square. Dragging the defining objects –the point O and the square $ABCD$ - indicates that the locus consistently appears to take such a form (as shown in Figure 5.1.3). The form can be analysed through classical geometrical methods by considering any edge of the square, and constructing the triangles formed by the line segment in its ‘variable’ position OP , and its ‘extreme’ positions OA and OB (as shown in Figure 5.1.4). The straightness –and half-size- of $A'MB'$ can be proved through analysis of similar triangles.



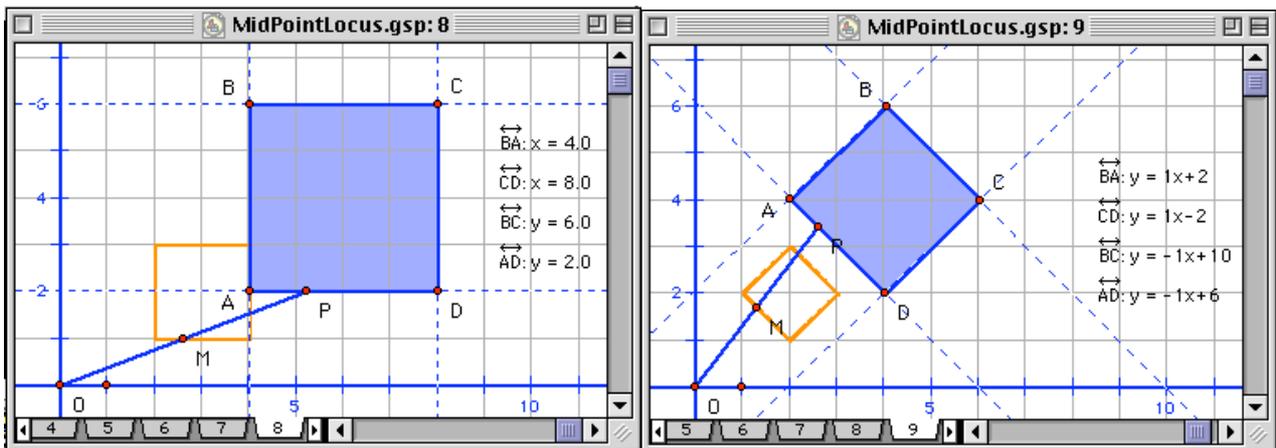
Figures 5.1.3-4: Dragging defining elements of the locus, and analysing it in classical terms.

In this light (and rather more directly) the relationship between the square on which P is moving and the shape formed by M can be conceived in transformational terms as a dilatation centred on O , with scale factor identical to the (given) ratio between the lengths of OP and OM (as shown in Figures 5.1.5 –before- and 5.1.6 –after).



Figures 5.1.5-6: Analysing the locus in transformational terms as a dilatation of the square.

Finally, the problem can be treated in co-ordinate terms. The fixed point is placed at the origin, and each of the edges of the square satisfies a co-ordinate equation for the corresponding straight line. The square may be positioned so that these equations are simpler in form (as shown in Figures 5.1.7) or more complex (as shown in Figure 5.1.8). In either case, the relationship between the equations of edges in the original square and its image can be examined.



Figures 5.1.7-8: Investigating the locus in co-ordinate terms.

Such an investigation points to a co-ordinate analysis of the relation. Although this should –at least initially- be conducted in relation to specific examples, it can be developed into a general analysis of the following form:

Take O as (0, 0), P as (p, q), and M as (m, n). The values of p, q, m and n vary as the points move. However, whatever the position of the points, the following can be said.

Since M is the midpoint of OP, $m = p/2$ and $n = q/2$, which is equivalent to $p = 2m$ and $q = 2n$.

If P runs along an edge satisfying equation $ax + by + c = 0$, then $ap + bq + c = 0$.

Replacing p and q in that result by their equivalents, $2am + 2bn + c = 0$.

Consequently, $am + bn + c/2 = 0$.

Hence whatever its position, the coordinates of M satisfy the equation $ax + by + c/2 = 0$.

This is the equation of a line parallel to the original edge on which P is moving, and half its distance from the origin.

5.2 *Midpoint Locus at Cams Hill*

At Cams Hill, work on *Midpoint Locus* was carried out with Year 10 top and Year 9 middle sets. A projected image was used to pose mathematical problems to the class for investigation by pupils working in pairs, and then to review their ideas.

Prompted by a display of the basic construction, at this stage without an animated trace (see Figure 5.1.1), pupils were asked to discuss and draw the locus of the mid point. Some pupils immediately commented that the locus would be a square, but when they were asked to describe the square all assumed that it would be the same as the one being traced by the end of the line. Teacher and class then discussed the key aspects of the square, the vertices, using the classical analysis (see Figure 5.1.4) to demonstrate how this may help pupils to construct the locus on paper. All pupils were then able to come up with a diagram of the locus.

The next task was to describe the locus, which gave rise to a number of interesting responses. Pupils first described the original as an enlargement of the locus; then moved to describing the locus as a reduction of the original. Pupils discussed the locus as being a quarter of the original before finally realizing that this was in area and not in length, and so shifting to describing the locus as half the size of the original.

Teacher and class then discussed an example where the square was positioned obliquely (see Figure 5.1.3) and each pupil produced a quick sketch and description of the locus that would occur.

The lesson then moved swiftly on to consider a square positioned on a coordinate grid (see Figure 5.1.7), with pupils sketching the locus and stating the coordinates of the vertices of the locus and the equations of the lines that would construct each edge of the locus. Pupils were by now confident of how the locus should look and were able to focus on accurately constructing it before reading the necessary coordinates and equations. A few pupils were then instructed to find a connection between the coordinates of the original and the corresponding coordinates of the locus and were able to see the connection between halving the size and halving the coordinates.

This was as far as teacher and classes managed to proceed within the lessons, but the teacher planned to extend the task in a future lesson to investigate the connection between not only the coordinates but also the equations of the edges.

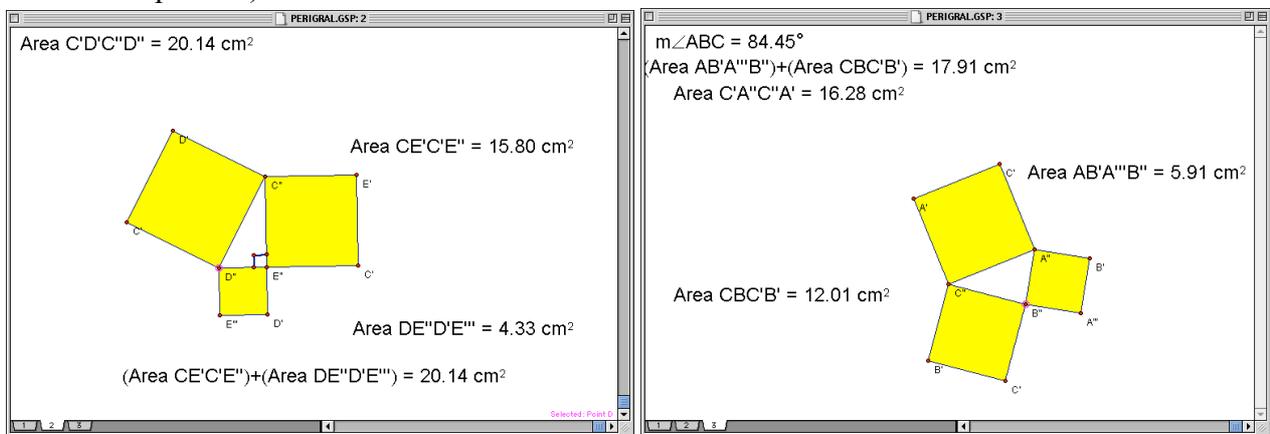
Reviewing the lessons, the teacher commented on how the DGS slides gave “good visual prompts to allow pupils to progress effectively through the task” and on how pupils “were able to link shape space and measure with algebra easily”. He was also pleased with the way in which “This approach allowed pupils to be instantly convinced of the outcome of such a movement and to focus on the task instead of worrying about the complicated description of what they were required to draw”.

6 Other activities

6.1 *Pythagoras Rule* at Neville Lovett

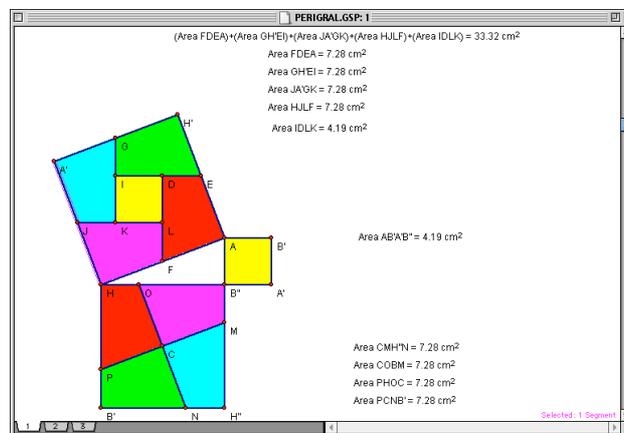
This work was carried out with a Year 9 second set over two lessons. Using an interactive whiteboard to introduce the topic to the whole class, the teacher employed DGS configurations to explain/show aspects of Pythagoras' Rule.

By dragging certain points on the first DGS configuration (Figure 6.1.1) the size of the right-angled triangle could be varied, with the display showing the corresponding area measurements. The teacher commented that this provided a model that could be easily moved in front of the class and that "Pupils can see (or are told) that the results are correct for every situation, and see visually the effects of moving parameters within the model". (When this lesson was discussed later, questions were raised as to why displayed results should be trusted, and how observed relationships could be explained).



Figures 6.1.1-2: Screensnaps of configurations for Pythagoras Rule.

In the second configuration, the constraint that the triangle be right-angled was absent. Again, by dragging certain points (Figure 6.1.2) the triangle could be varied, and areas computed and compared. (In the later discussion of this lesson, suggestions were made as how use of this template could be used to motivate the converse of Pythagoras Rule, and to bring out the functional relationship between the respective 'deficit' or 'excess' of the 'not always right' angle and the 'not always equal' areas.) Finally, although not used in the lessons, the teacher had prepared a dynamic model of Perigal's Dissection (Figure 6.1.3).



Figures 6.1.3: Screensnap of configuration for Perigal's Dissection.

6.2 The Baywatch Problem at Applemore

This activity took place over four lessons with a Year 9 top set. Pairs of pupils worked with a DGS on a tablet PC. The majority of them had used this technology on one previous occasion. For work with the whole class, a projected image was available.

The activity was based around the ‘Baywatch’ problem. *A person is in trouble in the sea. A lifeguard, standing at the water’s edge needs to come to the rescue. The person is 200 metres further along the shore and 100 metres out to sea. The lifeguard can run at 5 m/s and swim at 2 m/s. Find the optimum distance for the lifeguard to run along the shore, so that the person can be rescued in the shortest time.*

The four lessons were organised roughly as follows:

Lesson 1: Pencil & paper. Show how to set up DGS model

Lesson 2: Set up DGS model, try for initial values plus 1 or 2 other pairs

Lesson 3: Try a wider variety by tracing plotted point

Lesson 4: Class discussion and algebraic model

The teacher wanted the pupils to try the problem out with pencil and paper first to get a feel for the problem, but also to be able to appreciate the advantages of setting up a dynamic model. At the end of the first lesson she demonstrated how they would be setting up the DGS model in the next lesson so that they would be familiar with it (see Figure 6.2.1). She used a laptop and projector to do this. In the next lesson, some pupils were able to remember what they needed to do, which made it easier to work with the others.

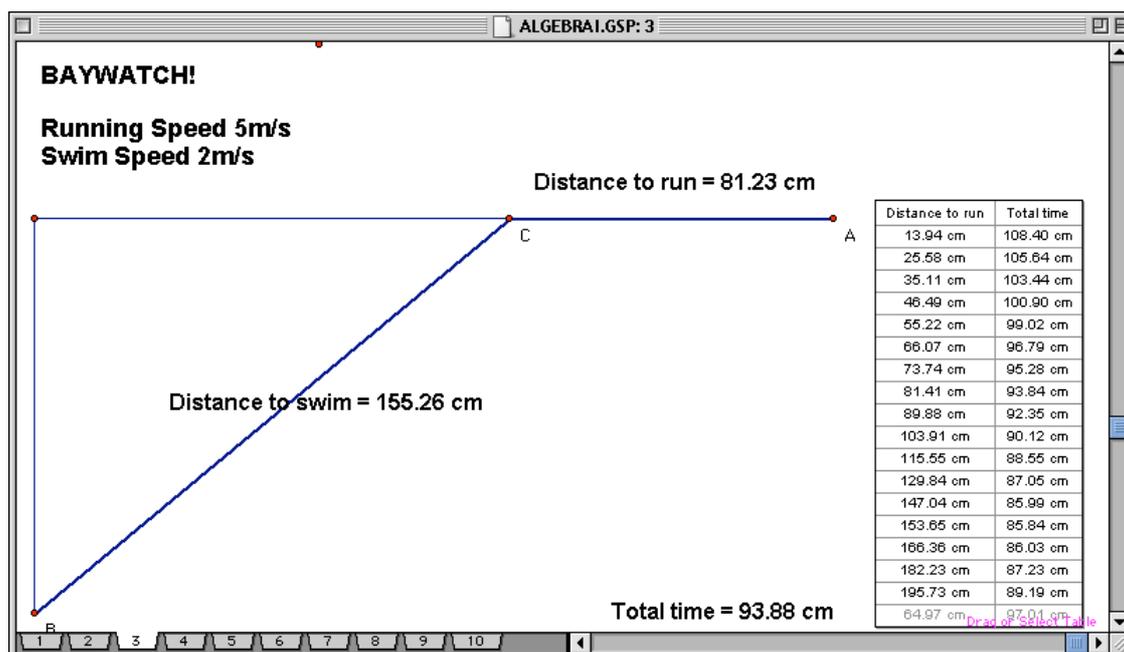
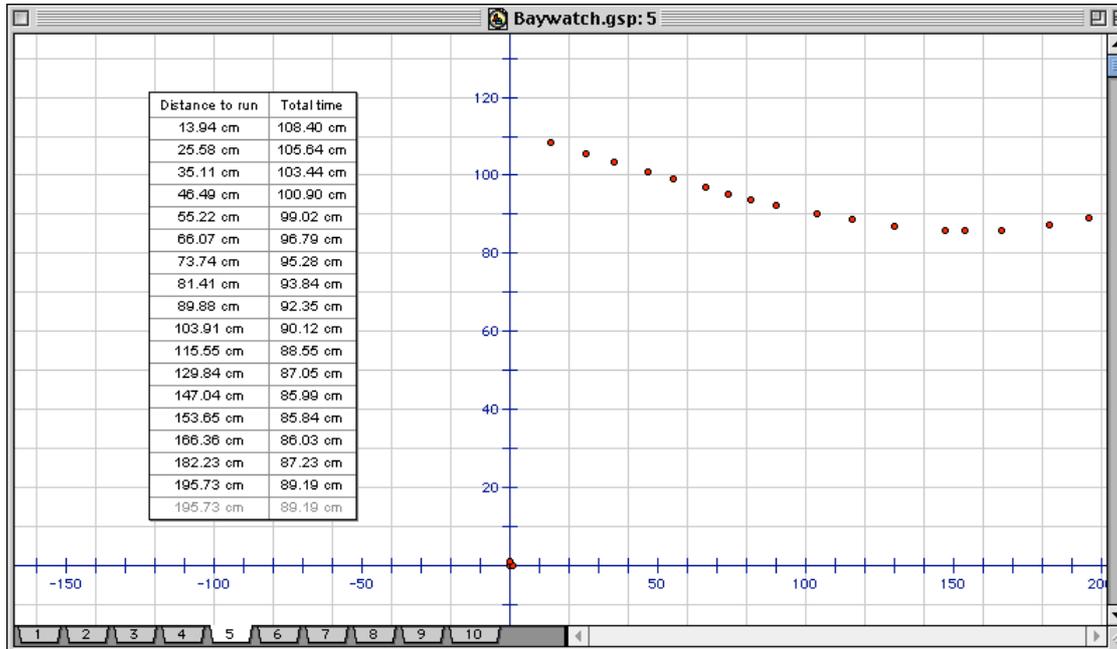


Figure 6.2.1: Screensnap of the DGS model for the Baywatch problem.

The teacher wanted the pupils to set the DGS model up for themselves so that they would understand exactly what each value was, so avoiding a ‘black box’ syndrome with values altering and the pupils not knowing for certain what each one represented. Although a few groups made mistakes with one or other of the calculations or measurements, without exception they were able to identify that they had made a mistake (by observing what happened to the values as they varied the running distance), and then to locate and rectify it. For example, one pair called the teacher over and said “Ours is wrong, the shortest time is obtained when swimming the diagonal”. They knew this was incorrect from doing the task with pencil and paper. They went on to say “We must have the swimming and running speeds the wrong way round; if someone could swim faster than they could run, the shortest distance would be best – and that’s the diagonal”. They then corrected it.

Pupils tabulated a range of values for the DGS model then inserted a new page as a copy of the first. They then hid the model, inserted a set of axes and plotted the points as a graph (see Figure 6.2.2). They were then asked to investigate a variety of swimming and running speeds (these would obviously vary with different life guards and also with different conditions; i.e. sea currents, dry or wet sand). They were asked to come up with some general guidelines or rules of thumb for lifeguards. Next, pupils were shown how to trace a plotted point, to avoid the need to tabulate data. Again, the teacher wanted pupils to have experienced the tabulating and plotting part of it so that they would understand what the plotted point was ‘doing’.



Figures 6.2.2: Screensnap of the DGS results graph for the Baywatch problem.

Pupils were then allowed to explore the problem and came up with a variety of rules; e.g.

- If the running speed is equal to or less than the swimming speed – swim the diagonal.
- For 2:5 enter the water about $\frac{3}{4}$ of the way along the shore.
- If the sand’s wet, so the running speed is slower, enter the water sooner.
- If the current is strong enter the water later.
- A few metres more or less running doesn’t make much difference to the overall time.

One group of pupils produced sets of traced graphs for different ratios and considered exploring the graph through the minimum points. The teacher discussed with pupils the idea of ratio of speeds; that the values 4 to 10 have the same ratio as 2 to 5, and would give the same optimum distance, but would halve the time. Finally the situation was modelled algebraically, and the corresponding graph superimposed onto their plotted points. The pupils then altered this for some of the other values of the parameters that they had tried, both plotting and tracing.

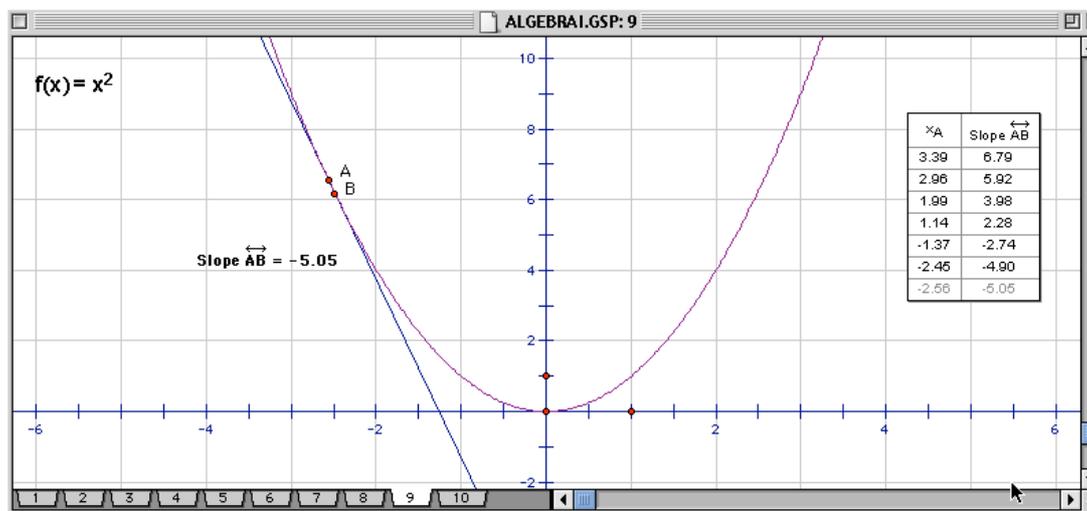
Reviewing the lesson, the teacher found this “definitely a beneficial approach”. Pupils appreciated the time saving aspect of using the software. It also allowed them to construct graphs quickly and easily. They had no problems understanding what they were doing. Questioned about the values displayed on their screens, all could all interpret the units; although all the displayed values were indicated as ‘centimetres’, the pupils knew which ones were actually metres and which ones were actually seconds. The teacher also commented on how the software allows pupils to explore the problem in far more detail and far more readily. She noted that pupils had no problems in making links between model and the graph and in grasping the final algebraic model and corresponding graph.

6.3 *Curve Gradients at Applemore*

This activity took place over four lessons with a Year 10 top set. Again, pairs of pupils worked with a DGS on a tablet PC. The majority of them had used this technology a number of times previously. For work with the whole class a projected image was available.

Pupils were asked to make a construction (see Figure 6.3.1) as follows:

- Plot the graph of a quadratic function;
- Construct two points on the resulting curve;
- Construct a line through the two points;
- Display the slope of the line.



Figures 6.3.1: Screensnap of secant line construction.

Using this construction, pupils were then asked to drag the two points so as to lie close together (even apparently on top of one other, but not coinciding exactly). The teacher commented that pupils could see, by selecting both points and moving them, how the slope of the line went from positive to zero to negative. Using this technique, pupils were asked to:

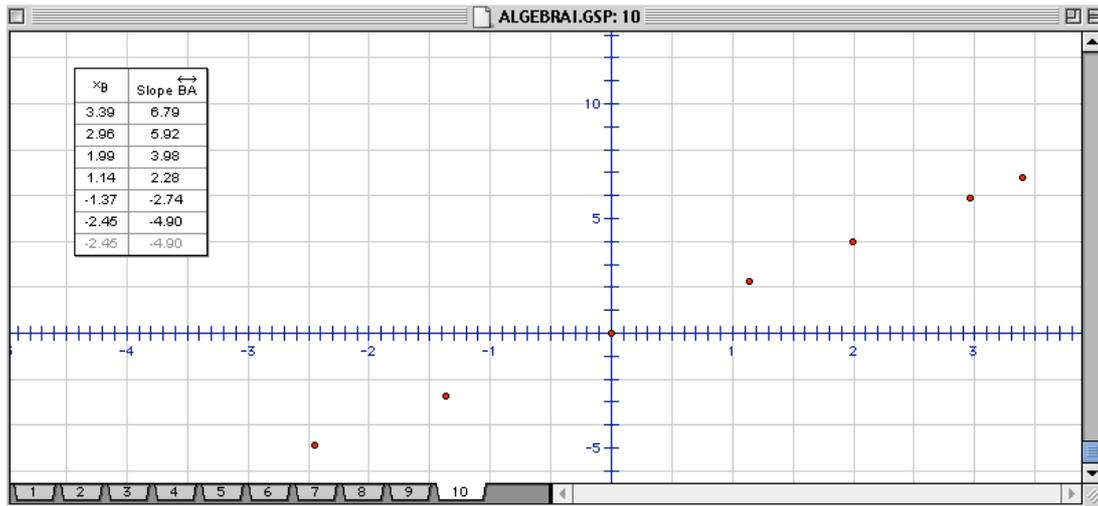
- Find the slope of the line at various positions on the curve;
- Tabulate the resulting (x -coordinate, slope) pairs of values.

The teacher commented that very accurate values could be obtained for the local slope of the curve in this way, whereas it was very difficult to get such accurate results by drawing tangents to a curve by inspection. (Note the difference between the pencil-and-paper tangent construction and the dynamic-geometry secant construction, and the potentially important issue of their mathematical relationship). The teacher saw the only disadvantage of using a DGS as being that pupils did not get experience of drawing tangents to curves, and she suggested that the next time she used this activity she might incorporate such pencil-and-paper work before switching to using the DGS. (Note that such a strategy might well accentuate for pupils the difference of techniques in the two tool systems).

Next, pupils were asked to create a new DGS sheet (see Figure 6.3.2) on which they were to:

- Plot the tabulated pairs on a graph;
- Construct a line of best fit through the resulting points;
- Find the equation of this line;
- Relate this equation back to the original situation.

The teacher commented on how, by plotting the points, it is easy to see that they lie on a straight line, although some pupils noticed the connection simply by looking at the tabulated values. Equally, the teacher reported that some pupils realised that translating the curve vertically would have no effect on its gradient. Finally, a few exceptional students went on to look at the ‘proof’ of differentiation, using the idea of a small increment tending to a limit.



Figures 6.3.2: Screensnap of tabulation and graphing of (ordinate, slope) pairs.

6.4 *Coordinate Designs at Cams Hill*

This activity was employed to introduce use of GSP on the Voyage-200 to low sets in Years 7 and 8 at Cams Hill. After a few simple instructions from the teacher, pupils were tasked to complete the drawing of a square from the two points and line segment already constructed. They were then asked to draw a triangle followed by a picture of their choice.

The teacher reported that pupils were immediately interested in using the Voyage-200 and “almost forgot that they were doing ‘work’”. Pupils were not hindered by drawing the axes or held back by inaccurate plotting of points. The ease of deleting and moving points and segments also allowed pupils to change and modify pictures with ease. Pupils enjoyed working on the task and made excellent progress. By the end of the lesson the task had been extended to “Who can draw the best car?” allowing pupils to include circles in their drawings as well as points and line segments. The plenary involved each group displaying its work to the rest of the class, using a projector. The teacher reported that the pupils loved this.

7 Overview and conclusions

Phase 2 of the study successfully initiated work to develop the Phase 1 material into viable and valuable teaching and learning activities. The small-scale classroom work which it was possible for teachers to carry out within the limited timescale available has confirmed the promise of the selection of activities trialled (from amongst the larger collection developed in Phase 1). In particular, many of these produced a valuable degree of interplay between algebraic and geometric concepts and techniques, and stimulated mathematical argumentation by pupils.

One noteworthy feature of many of these activities is the way in which they could be adapted so as to be accessible to pupils of a wide range of prior attainment. Another is the way in which they lent themselves to prediction and testing, and more broadly to mathematical discussion and argumentation. The possibilities of publicly projecting images, of manipulating and modifying dynamic figures and diagrams, and of generating and representing associated data, all played an important part in stimulating and supporting class discussion.

On those occasions when pupils were given opportunities to work independently with DGS, within a clear structure provided by the teacher, they proved themselves capable of learning necessary techniques and producing appropriate constructions. Nevertheless, many of the teachers reported that they themselves were relatively unfamiliar with DGS as were their pupils; under these circumstances, there was a preference for relatively short and simple use of prepared DGS templates managed by the teacher.

Much remains to be done in order to develop these activities into a form which will enhance mathematical argumentation within the parameters of the current National Curriculum and permit a genuine integration of DGS as new mathematical technologies. In particular, there is a need to develop units of work which incorporate more extended and sustained use of DGS, allowing more systematic development of necessary technique, deeper analysis of mathematical situations, and fuller elaboration of mathematical lines of argument than is possible within a single, one-off lesson or lesson starter.

8 Acknowledgments

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As well as funding from QCA, we acknowledge additional support as follows:

Texas Instruments UK provided a set of 12 TI Voyage-200 hand-held computers (together with LCD displays for OHPs) for use by teachers in the project – these have both *Cabri Geometry II Plus* and *the Geometer's Sketchpad* Version 4 already installed.

The 'Teachers Teaching with Technology' (T-cubed) programme provided support for 12 teachers using the Voyage 200 equipment with Adrian Oldknow as course tutor.

Chartwell-Yorke provided a site-licence for the Geometer's Sketchpad to one of the schools.

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