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INSTRUMENTING MATHEMATICAL ACTIVITY: REFLECTIONS ON KEY STUDIES OF THE EDUCATIONAL USE OF COMPUTER ALGEBRA SYSTEMS

ABSTRACT: This paper examines the process through which students learn to make functional use of computer algebra systems (CAS), and the interaction between that process and the wider mathematical development of students. The result of 'instrumentalising' a device to become a mathematical tool and correspondingly 'instrumenting' mathematical activity through use of that tool is not only to extend students' mathematical technique but to shape their sense of the mathematical entities involved. These ideas have been developed within a French programme of research –as reported by Artigue in this issue of the journal- which has explored the integration of CAS –typically in the form of symbolic calculators- into the everyday practice of mathematics classrooms. The French research –influenced by socio-psychological theorisation of the development of conceptual systems- seeks to take account of the cultural and cognitive facets of these issues, noting how mathematical norms –or their absence- shape the mental schemes which students form as they appropriate CAS as tools. Instrumenting graphic and symbolic reasoning through using CAS influences the range and form of the tasks and techniques experienced by students, and so the resources available for more explicit codification and theorisation of such reasoning. This illuminates an influential North American study –conducted by Heid- which French researchers have seen as taking a contrasting view of the part played by technical activity in developing conceptual understanding. Reconsidered from this perspective, it appears that while teaching approaches which 'resequence skills and concepts' indeed defer –and diminish- attention to routinised skills, the tasks introduced in their place depend on another –albeit less strongly codified- system of techniques, supporting more extensive and active theorisation. The French research highlights important challenges which arise in instrumenting classroom mathematical activity and correspondingly instrumentalising CAS. In particular, it reveals fundamental constraints on human-machine interaction which may limit the capacity of the present generation of CAS to scaffold the mathematical thinking and learning of students.

KEYWORDS: Computer algebra systems; Computer and calculator uses in education; Mathematical thinking; Mathematics teaching and learning; Secondary and post-secondary education.

INTRODUCTION

This paper originated as a response (Ruthven, 2001) to a presentation on 'instrumentation and the dialectics between conceptual and technical work' made by Michèle Artigue (2001) at the second CAME (Computer Algebra in Mathematics Education) symposium. Our revised versions appear together in this issue of the journal. At the same symposium, Kathy Heid (2001) made a presentation on 'theories that inform the use of CAS in the teaching and learning of mathematics', commenting in passing that 'theories relating to... instrumentalization are becoming increasingly important' (p. 2). As editor of *Educational Studies in Mathematics*, I was also aware of a recently published paper in which Lagrange (2000/01) raised some important points of difference between the conceptualisation of technique within the French programme of research (Artigue, 1997; Lagrange, 1999; Guin & Trouche, 1999; Trouche, 2000) as against earlier North American studies seeking to 'resequence skills and concepts' (Heid, 1988; Judson, 1990; Palmiter, 1991). These issues were alluded to but not pursued in any depth within the symposium. Consequently, in revising my response for publication, I decided that I should incorporate a discussion of them.

THE INTERPLAY BETWEEN INSTRUMENTED SCHEMES AND MATHEMATICAL NORMS

The conceptual framework employed by the French research (Artigue, 2002)[1] seeks to take account of both the cognitive and cultural issues which arise in integrating use of CAS –typically in the form of symbolic calculators- into the everyday practice of mathematics classrooms.

From a cognitive perspective, following Verillon & Rabardel (1995), the French researchers focus on the process by which students 'instrumentalise' a calculator to become a mathematical tool, and correspondingly 'instrument' mathematical actions through use of the tool; a process "involving the construction of personal schemes or, more generally, the appropriation of social pre-existing schemes" (Artigue, 2002: §II.2). The cognitive systems identified by the research reveal subtle relations. For example, "local schemes… for dealing with specific tasks [such as] the determination of limits in +∝, interact with more global schemes [such as]… using a specific [calculator] command… to get an approximate value for a given number, whilst working in… 'exact' mode" (§III.1). Likewise, this latter 'scheme of approximate detour' "can take different meanings, according to the student and the situation" (§III.1); in effect, it can be coordinated with other schemes as "part of a process of anticipation or verification", or "as a substitute for [an] intended result", or "as a stratagem in which the peculiarities of the approximate value which is obtained are used for guessing the exact value" (§III.1).

A central theme of the French research is the complexity of the process of 'instrumental genesis' through which an initially unfamiliar technological artefact becomes a functional mathematical instrument for students. This is shown to involve a trajectory in which the analytic medium (and corresponding calculator mode) privileged by students progressively shifts from the graphic to the symbolic:

> Instrumental genesis involves an evolution in the roles of the different applications of the calculator… [D]uring the first phase of this genesis, the graphical application plays a predominant role both in exploration and solving; the Table application plays a control role; the symbolic application HOME plays a marginal role (essentially the computation of the derivative). In a second phase, even if the graphical application still predominates, HOME becomes more involved in the computation of exact values for the function and its derivative, for calculating limits, or checking some graphical results, with a role of support to the graphical work. In the third phase -the so-called calculus phase- an inversion takes place: the symbolic application becomes the predominant tool in the solving process, jointly with paper & pencil work, while the graphical application becomes mainly an heuristic tool, for anticipation and control. (Artigue, 2002: §III.2)

Equally, however, this evolution is linked to a shift in the mathematical norms operative at different stages of the programme that the students are following. Adopting an anthropological approach, attributed to Chevallard, the French researchers analyse such cultural norms in terms of the types of task posed, the techniques approved for accomplishing these tasks, and the technical and theoretical discourses accepted for conveying these techniques and sanctioning their use (Bosch & Chevallard, 1999). This conceptualisation is closely related to a model of classroom practice:

> In a French classroom today, work on a new mathematical theme begins with a first phase of exploratory work, after which certain paper & pencil techniques are identified and become 'official'. Students are then trained to use these techniques in different contexts and some routinisation work takes place. A discourse of a more theoretical nature goes along with this institutionalisation of official techniques, having explicative and justificative aims. (Artigue, 2002: §IV)

Consequently, the French research has been particularly sensitive to the breakdowns in the functioning of this model observed when calculators are introduced to classrooms. One factor is the proliferation of calculator-mediated techniques in the absence of officially recognised forms, inhibiting clear lines of collective development:

> There is a temptation to allow students to discover up to what point the calculator can enrich their solution strategies, and this can result in a non-productive technical overload. When the educational institution does not give teachers any rule for selection, they are less sensitive to the necessity of making choices, and not equipped to make these in a rational way. This leads to an explosion of techniques which remain relatively *ad hoc,* and pose a didactic obstacle to the progressive building of mathematical activity instrumented in an efficient way. (Artigue, 2002: §IV)

Another factor is the absence of an accepted discourse for calculator-mediated techniques, and the difficulty of devising such a discourse without going beyond what is conventionally viewed as mathematical:

> A discourse has to be constructed for instrumented techniques. Once more, difficulties are obvious because this discourse will call up knowledge which goes beyond the standard mathematics culture. It will necessarily intertwine standard mathematics knowledge and knowledge about the artefact. (Artigue, 2002: §IV)

A final and related factor is the difficulty of establishing a status for calculator-mediated techniques, while these are not recognised by the larger institution of statutory school mathematics, and remain on the margins when classroom mathematics is codified:

> The institutionalisation process… essentially dealt with knowledge relevant to mathematical work in the paper & pencil environment. These characteristics did not help instrumented techniques to gain mathematical status… Even when fully legitimated, they remained at a sort of intermediate status in the classroom culture, for which Defouad introduced the notion of 'locally official techniques'. (Artigue, 2002: §IV)

Such observations led the French researchers to appreciate the significant interplay between mathematical norms and instrumented schemes, and this informed their development of teaching designs which aimed to more actively manage the process of instrumental genesis.

THE SOCIO-PSYCHOLOGICAL DEVELOPMENT AND EPISTEMIC SOURCES OF CONCEPTUAL SYSTEMS

The French research is grounded in a constructivist tradition which insists on attending to the socio-psychological development of conceptual systems, not just to their codified disciplinary form:

> [T]he psychological definition of a concept cannot be reduced to its scientific definition… [M]athematicians normally strive to be precise, complete and parsimonious when they write definitions, whereas psychologists try to understand how concepts are progressively shaped, by different kinds of situations and competences and by different kinds of linguistic representation and symbols. (Vergnaud, 1997: p. 5)

From this cognitive perspective, even the meanest competence depends on a corresponding scheme, and those forms of action and expression typically deemed mathematical depend upon hierarchies of schemes upon schemes upon schemes. In particular, it is schemes which sustain the categorisation and relational organisation which are usually taken as constituting conceptualisation in the cognitive sense, even if they remain implicit in exercising competence.

For example, the development of 'framing schemes' for calculator graphing depends both on the tasks that students are exposed to, and on the related mathematical schemes available to them. To take the case of Frederic (Artigue, 2002). At the time of his first interview he –like other students- is "familiar with polynomial functions essentially" (§II.3). Asked to study the variation of a function of an unfamiliar type, having two branches, one falling well outside the standard graphing window, he frames a graph containing only the visually available branch (Fig. 3). This has many prototypical features of a polynomial (specifically cubic) graph, with two extrema visible and clear trends to each side, where the function apparently rises gently to the left and falls away more steeply to the right. It seems, then, as if Frederic may be adapting a specific visual scheme already available to him to guide his framing of the graph, employing this in combination with more classic -and less technologically specific- framing strategies such as modifying the range of values displayed in the graphing mode. Equally, when Frederic triangulates this model of the situation by following an established technique of producing and factorising the symbolic derivative, he has a relevant scheme available, established on the basis of his prior experience of derivatives taking the form of "a second degree polynomial, or possibly a fractional expression whose denominator is a square and numerator a first or second degree polynomial" (#19), but proves unable to adapt it to a result which features a third degree polynomial in the numerator of the fractional form. This suggests that the schemes available to Frederic at this moment for the analysis of variation remain rather specialised ones, reflecting the narrow range of functions he has encountered, although open to some degree of adaptation. Evidence of continuing specificity in Frederic's developing variation schemes, even at the end of the year when it is suggested that he has "developed specific and efficient instrumented schemes for framing and variation analysis, by connecting the symbolic and graphical applications of the calculator" (§III.2), comes from his difficulties in analysing "a new type of function, mixing square roots and trigonometric functions, generating new phenomena linked to the discretisation processes used by the calculator" (§III.2). This not only illustrates the complex development of framing schemes, but suggests that building a coherent conceptual system and an overarching concept of framing involves the progressive coordination of many rather specific schemes.

Within the culture of mathematics, however, it is precisely such overarching concepts which are usually singled out as the core of a mathematical system; typically, the organising and unifying ideas underpinning some compact, abstract codification of the system. This has encouraged an overestimation of the significance of disciplinary 'big ideas' and 'grand theory' in thinking, learning and teaching:

> Presumptions about the diversity and grain size of knowledge involved in mathematical and scientific expertise have typically been too few and too large. Traditional analyses of expert reasoning have focused [on] the use of powerful, general pieces of core knowledge... [But] knowledge is distributed across a far greater number of interrelated general and context-specific components than either those analyses of expertise or textbook presentations suggests... Formal descriptions of knowledge in mathematical and scientific disciplines have... exerted a strong influence over what knowledge we want students to learn and how we expect them to learn it. (Smith, diSessa & Roschelle, 1993/94: 145-5).

Moreover, even in disciplinary terms, powerful mathematical systems typically exploit several epistemic sources, which might even be described as component senses inasmuch as they draw on different source metaphors. For example, adapting Thurston (1994: p. 163), the ways of conceiving the derivative of a function within school mathematics include:

- Infinitesimal ratio: as the ratio ('gearing') between an infinitesimal change in the argument and the resulting change in the function.
- Instantaneous rate: as the instantaneous rate of change ('speed') of the function.
- Tangent slope: as the slope of a line tangent to the graph of the function.
- Linear approximation: in terms of the line best fitting the function nearby a point.
- Microscopic image: in terms of the limiting graph of the function nearby a point under a microscope of higher and higher power.
- Symbolic correlate: in terms of x^n to nx^{n-1}, $\sin x$ to $\cos x$, etc.

Coordinating these different senses plays an important part in creating a multifaceted conceptualisation of derivative, and this involves the development of a unifying technical and theoretical discourse, and an associated technical apparatus for translating between senses.

Moreover, new tools permit a rebalancing of senses, or reorganisation of a particular sense. For example, the availability of calculating tools facilitates a numeric treatment of derivative in terms of the infinitesimal ratio sense; likewise, graphing tools assist work on the tangent slope and linear approximation senses. Equally, the new possibility of 'zooming' in and out on a graph permits an operationalisation of the microscopic image sense through tasks and techniques focusing directly on the local slope of the graph itself, without appeal to secant and tangent constructions (Tall, 1996). Similarly, the use of CAS symbolic routines permits a reframing of elementary problems of function variation in terms of local power series without the need to appeal to a prior idea of symbolic derivative, but with the potential to motivate its subsequent development (Ruthven, 1995).

In effect, then, our sensation of mathematical entities, our very impression of interacting with them, is grounded in the systems of task and technique which invoke them, is constituted by the perceptual and operational schemes which these evoke, and is shaped by the mediating tools and moderating discourses which frame these. Artigue highlights the epistemic function of techniques in evoking this mathematical world:

> I would like to stress that techniques are most often perceived and evaluated in terms of *pragmatic value*, that is to say, by focusing on their productive potential (efficiency, cost, field of validity). But they have also an *epistemic value*, as they

> contribute to the understanding of the objects they involve, and thus techniques are a source of questions about mathematical knowledge. (Artigue, 2002: §II.1)

However, Artigue also points out that, as they are routinised:

> techniques lose their mathematical 'nobility' and become simple acts. Thus, in mathematical work, what is finally considered as mathematical is reduced to being the tip of the iceberg of actual mathematical activity, and this dramatic reduction strongly influences our vision of mathematics and mathematics learning and the values attached to these. (Artigue, 2002: §II.1)

There is, nevertheless, an important relationship between socio-psychological and disciplinary perspectives. Students develop cognitive structures well adapted to the cultural norms framing classroom mathematics, in the form of the situations they encounter, the resources available to them, and the expectations conveyed to them. If tasks are strongly compartmentalised, techniques highly prescribed, and discourses severely restricted, then the mathematical senses and cognitive schemes that students develop will be correspondingly fragmented and inflexible. Within the field, it is widely agreed that teaching and learning should aim to build the coordinated mathematical senses and cognitive structures which constitute the two faces of conceptual understanding.

THE DIALECTIC BETWEEN TECHNICAL ACTIVITY AND CONCEPTUAL UNDERSTANDING

An important feature of the French research on CAS has been a deepening reservation about what may have become too received a distinction -even an opposition- between technical activity and conceptual understanding; and too strong an assumption that access to computational technology permits a reduction -even an elimination- of the former in favour of the latter. In particular, Lagrange (2000/01) has challenged an influential strand of North American work which has sought to 'resequence skills and concepts' through technology-supported teaching approaches. He argues that this aspiration embodies "a misunderstanding regarding the possibility of devising a teaching approach aimed directly at concepts" (p. 11). Whereas, the French researchers give technique "a wider meaning than is usual in educational discourse" (Artigue, 2002: §II.1), comprising not just recognised routines for standard tasks but more "complex assembl[ies] of reasoning and routine work", the American research delimits the technical domain more narrowly in terms of "routine manipulations", "computational procedures" and "algorithmic skills" (Heid, 1988). These are more than simple differences of demarcation and terminology. For Lagrange:

> The opposition between concepts and skills masks an essential point. There is a technical dimension to the mathematical activity of students which is not reducible to skills. When technology is used, this dimension is different, but it retains its importance in giving students understanding. (Lagrange, 2000/01: p. 14)

In developing this point, Lagrange uses Heid's (1988) study of an applied calculus course as his principal foil. This fastidious and illuminating study appraised a CAS-supported teaching approach in which the development of student "mastery of the usual algorithmic skills of computing derivatives, computing integrals, or sketching curves [in the pencil-and-paper medium]" (p. 4) was deferred to the closing phase of the course. The guiding hypothesis was that:

> If mathematics instruction were to concentrate on meaning and concepts first, that initial learning would be processed deeply and remembered well. A stable cognitive structure could be formed on which later skill development could build. (Heid, 1988: p. 4)

Accordingly, during the first twelve weeks of the course, experimental classes "used the microcomputer to perform most of the algorithms on which students in the regular sections of the course spent most of their time" (p. 6), allowing "concepts [to be] developed in greater depth and breadth and through a greater variety of representations than in the traditional class" (p. 7). In the final three weeks of the course, by contrast, the experimental classes "covered only traditional algorithms and procedures with little attention to applications or to problem solving" (p. 6).

For Lagrange, however, this focus on skills misses the broader part played by technique:

> Of course, at certain moments a technique can take the form of a skill. This is particularly the case when a certain 'routinisation' is necessary... It is certain that the availability of new instruments reduces the urgency of this routinisation. Heid's study confirms this. But techniques must not be considered only in their routinised form. The work of constituting techniques in response to tasks, and of theoretical elaboration on the problems posed by these techniques remains fundamental to learning. (Lagrange, 2000/01: p. 16)

From this perspective, technique -whether mediated by technology or not- fulfils not only a pragmatic function in accomplishing mathematical tasks, but an epistemic function in building mathematical concepts.

Indeed, Heid's (1988) study illustrates contrasting approaches to this constitution of techniques and to their theoretical elaboration. The comparison group followed the "traditional large lecture for the same course" (p. 5), where constitution of techniques took the form of "concentration on skills and algorithms throughout" (p. 7). In particular, "assignments, tests and quizzes dealt mainly with the execution of traditional skills" (p. 6) although "a small number of concept-oriented questions were included on the tests for purposes of comparison with the results from the experimental classes" (p. 6). Such theoretical elaboration as did occur seems to have "focus[ed] on a symbolic or formulaic representation of concepts" (p. 9), and to have fallen primarily to the lecturer who "taught algorithmic skills as the calculus concepts were being developed" (p. 6).

In the experimental classes, the constitution of a quite different system of techniques appears to have played an important part. The shift to "reasoning in nonalgebraic modes of representation [which] characterized concept development in the experimental classes" (p. 10) not only created new types of task, but encouraged systematic attention to corresponding techniques. For example, Heid (1988) reports that, during the main phase of the experimental course, "[t]he students interpreted the properties of graphs in applied situations, noting, for example, the meaning of *y*-intercepts, of differences in *y*-values, and of intersection points" (p. 8). These students performed correspondingly better on 'conceptual' assessment items incorporating such tasks, such as one concerning "G(*t*)... the number of people unemployed in a country *t* weeks after the election of a fiscally conservative President" which called for translation of "facts about the graph *y*=G(*t*)" such as "[t]he *y*-intercept of *y*=G(*t*) is 2,000,000" and "G(20)=3,000,000" into "statements about the unemployment situation" (p. 19). Not only did the 'conceptual' phase of the experimental course expose students to such wider techniques; it also appears to have helped students to develop proficiency in what had become standard tasks, even if they were not officially recognised as such, and had not been framed so algorithmically, taught so directly, or rehearsed so explicitly as those deferred to the final 'skill' phase.

Equally, "the experimental classes were the only small-class sections of the course offered that semester" (p. 5). This facilitated use of a pedagogy in which "[c]lass discussions and problem sets focused attention on executive decision making in problem solving rather than on the actual execution of standard computational procedures" (p. 6). Thus there was considerably greater opportunity for students in the experimental classes to become actively engaged in theoretical elaboration. Interviews found corresponding evidence of greater discursive resources and facility on the part of students from the experimental classes, who "gave a broader array of appropriate associations when explaining the concept of derivative" compared with "responses from the students in the comparison class, [which] although not inaccurate, were lacking in the detail provided by the students in the experimental classes" (p.15). Similarly, students in the experimental classes "verbalized connections between [concepts]" (p. 17) and reasoned from more basic principles to repair mistakes or fill gaps which emerged in their interview responses (pp. 15-16).

In its own terms, then, the experimental teaching approach studied by Heid did indeed succeed in 'resequencing skills and concepts'. It deferred routinisation of the customarily taught skills of symbolic manipulation until the final phase of the applied calculus course, while the attention given to a broader range of problems and representations in the innovative main phase supported development of a richer conceptual system. Equally, however, in the terms of the French theory, this conceptual development grew out of new techniques constituted in response to this broader range of tasks, and from greater opportunities for the theoretical elaboration of these techniques. At the same time, standard elements emerged from these new tasks, characteristic of the types of problem posed and the forms of representation employed, creating a new corpus of skills distinct from those officially recognised.

Finally, just as very different mathematical worlds were evoked in the experimental and traditional courses compared within Heid's study, similarly important institutional differences emerge between the characterisation of calculus in these North American courses and those studied in the French research. Indeed, what Artigue has written of the contrast between school and university mathematics in France might be adapted to summarise all these comparisons: that "beyond the common vocabulary and the apparent similarity of tasks and techniques, [they] develop profoundly different relationships for common mathematical objects" (Artigue, 1999: p. 1378). In marked contrast to either version of the American applied calculus course, the tasks prominent in the French research reflect a preoccupation with anomalies, often depending on contrived or unfamiliar situations. This helps to explain why the complexities of instrumentation emerge much more strongly in the French work: such phenomena are foregrounded by the use of tasks which deliberately take students beyond their curricular experience and place them in situations devised to expose limitations of the schemes available to them. For example, students are confronted not just with a standard monster of analysis (Artigue, 2002: §V.1), but with functions "of a type not familiar to [them]" for which "the graph obtained in the standard window... [is] far from being accurate" (§II.3), or such that "the small size of [the leading] coefficient makes... the graphical representation of the function in the default calculator windows incompatible with the[ir prior] 'algebraic' study" (§III.1).

THE INTERPRETIVE LIMITS OF HUMAN-MACHINE INTERACTION

More immediately, the French research also highlights fundamental constraints on human-machine interaction which impose particular limitations on the use of the present generation of CAS to scaffold the mathematical thinking and learning of students.

Artigue reports how the French research has explored use of the CAS as "a lever to promote work on the syntax of algebraic expressions… as something necessary for efficient communication with the machine" (2002: §V.2). Such exploration highlighted the importance of students developing the largely tacit instrumentation schemes routinely employed by expert CAS users. For example, Artigue (1997) reports an episode in which younger students -relatively inexperienced in using CAS- were working on a task aimed at engaging them with "work on bracketing and the writing of fractions" with a view to "constructing a situation in which the formal constraints of writing and notably of bracketing will appear to the students as [the adidactic] constraints" (p. 147) of mastering operation of the machine. The students were asked to translate algebraic expressions written in spatial form -such as $a+\frac{2}{5}-(\frac{a+2}{5})$- into a corresponding linear form -suitable for entry into the CAS as a string of keyboard characters- and were deemed to have succeeded if and when this resulted in the original expression being displayed on the CAS screen. Some students, "following the instructions scrupulously" (p. 149), tried to reproduce on the screen the exact bracketing of the original. They found that entering the strings *a*+2/5-(*a*+2)/5 and *a*+2/5-((*a*+2)/5) into the CAS produced the same screen expression, $a+\frac{2}{5}-\frac{a+2}{5}$, from which all brackets had been stripped out; whereas entering *a*+2/5-[*a*+2]/5 produced $a+\frac{2}{5}-\frac{[a+2]}{5}$ which retained the square brackets surrounding only the numerator. Moreover, whereas applying the simplify command to the first of these screen expressions produced a single fraction $\frac{4a}{5}$, applying the same command to the second yielded no such reduction, but rather $[-\frac{a+2}{5}]+a+\frac{2}{5}$.

As Artigue (1997) notes, "it was for the weakest students that the[se] discrepancies between the semiotic characteristics of the two environments were most perturbing" (p. 150), reflecting their lack of resources to build schemes through which CAS results could be interpreted and adapted mathematically. This example signals some of the complexities surrounding the instrumentation of mathematical actions by means of a CAS. Success in managing the 'pseudo transparency' of shifts between the differing representation systems employed at the CAS interfaces (such as between linear and spatial representation of expressions) depends on users' capacities to translate between mathematically equivalent but semiotically distinct forms (p. 149). Similarly, success in managing issues of 'double reference' where the conventions of the machine diverge from those of school mathematics (such as the emergence of square brackets as mathematical and computational 'faux amis') depends on alertness to critical differences in the ways in which symbols are used and operations implemented (p. 152). The invisibility of such issues to mathematically expert and computationally experienced CAS users reflects the availability to them of a complex system of instrumentation schemes fusing mathematical action with use of the machine.

Artigue (2002) also makes the point that while symbolic processing by CAS may be highly efficient in pragmatic terms, in epistemic terms "[the] result can look very different from the initial expression, and the student is in quite a different situation from paper & pencil work where, simplifying step by step, she knows the different intermediate expressions she has produced" (§V.2). Artigue also points to the way in which the "routinisation [of techniques] is accompanied by a weakening of the associated theoretical discourse and by a 'naturalisation' or 'internalisation' of associated knowledge which tends to become transparent, to be considered as 'natural'" (§II.1). Both automation and routinisation, then, tend to curtail -even suspend- the kinds of accounting for mathematical actions which underpin normal classroom communication. Often, a sufficient account is provided by the component inscriptional actions of the technique - such as drawing and marking up diagrams, or performing written computations and manipulations- with the flow between them indicated by the spatial organisation of written presentation and the temporal sequencing of oral commentary. The adequacy of such an account depends on its audience being able to bring to bear schemes which enable them to recognise -and usually elaborate- the component actions, so enabling them to identify an overall flow of activity and to fill gaps in the material presented. Where task and/or technique are not standard for the audience, it becomes more necessary to fill out the basic inscriptional record by providing some degree of commentary, explaining individual actions and their place in an overall organisation. Emergent discrepancies of interpretation can be addressed through dialogue between presenter and audience, thus elaborating, negotiating and refining the account sufficiently for it to become mutually intelligible and acceptable. Such accounting underpins the interactive processes of appropriation which play such an important part in classroom teaching and learning (Newman, Griffin & Cole, 1989). In effect, it is through such accounts that cognitive structures and cultural systems are coordinated.

However, the interaction of users with machines takes a more limited –and asymmetric- form:

> [I]nteraction between people and machines requires essentially the same interpretive work that characterizes interaction between people but with fundamentally different resources available to the participants. In particular, people make use of a rich array of linguistic, nonverbal and inferential resources in finding the intelligibility of actions and events, in making their own actions sensible, and in managing the troubles in understanding that inevitably arise. Today's machines in contrast, rely on a fixed array of sensory inputs, mapped to a predetermined set of internal states and responses. The result is an asymmetry that substantially limits the scope of interaction between people and machines. (Suchmann, 1987: pp. 180-1)

The French research provides a specific example in an episode in which students -relatively experienced in using CAS- were charged with the task of transforming the trigonometric sum, $\sin(x)+\sin(2x)$, into a trigonometric product (Lagrange, 1999: p. 5). In response to a programmed command, the CAS gave the expression $2\sin(\frac{2x+x}{2})\cos(\frac{2x-x}{2})$. The students wanted to simplify this to $2\sin(\frac{3x}{2})\cos(\frac{x}{2})$. To their surprise, the response of the CAS to repeated simplify commands was first to give the original expression and then $\sin(x)+2\sin(x)\cos(x)$. To the students, aware of the overarching goal, the emergent subgoal of simplifying the subsidiary algebraic expressions $2x + x$ and $2x - x$ was clear. To them, this was transparently the sense of the command to simplify. In other words, their articulation of the simplification operation was a situated one. But, of course, no model of the larger task -and no situated sense of the command- was available to the CAS; it was unable to take account of the wider mathematical context giving rise to the instruction. The machine is unable to interpret or adapt an instruction to accord with the wider purpose so evident to a user; it can only operate literally, either in terms of the formal elections made by the user, or of preset defaults –which, as in this case, may fail to coincide serendipitously with the wider purpose of the user. The effective instrumentation of mathematical reasoning by means of a CAS depends, then, on precise reframing of situated purposive actions into the decontextualised formal register of the machine, and a corresponding reframing of results.

CONCLUSION

Artigue (2002: §V.3) identifies two major ways in which using CAS can contribute to the development of mathematical knowledge.

The first of these constitutes the distinctive element of the French position. It is that instrumenting graphic and symbolic reasoning through using CAS influences the range and form of the tasks and techniques experienced by students, and so the resources available for more explicit codification and theorisation of such reasoning. Viewed optimistically, through mastering and elaborating instrumented techniques students develop an operational facility with important components of the conceptual system. This is a persuasive argument in principle, but it does invite certain questions of practice. In some respects, effective independent use of a CAS appears to impose greater technical and conceptual demands on students (as illustrated in the previous section): not only does specifying CAS operations require mastery of a sometimes complex formal register; interpreting the (didactically uncontrolled) results of CAS operations is likely to take students beyond those mathematical forms institutionally recognised at their particular stage of the curriculum. The argument, then, is somewhat double-edged, particularly while the institution of school mathematics does not valorise CAS-instrumented techniques in their own right. Under such circumstances, the costs and challenges of instrumented technique may not bring sufficient institutional benefits to be persuasive to teachers and students.

Rather, the greater incentive may be to develop a carefully controlled student experience of CAS with a view to avoiding 'unhelpful' difficulties and anomalies, an approach dependent on greater structuring by the teacher or instructional resources. At the most structured extreme, it may not be unusual to find detailed 'keystroke by keystroke' instructions being given to students. As use of the tool moves towards this extreme, it becomes a means of bypassing –rather than mastering- technical demands, correspondingly sacrificing the conceptual benefits of instrumentation. Yet, the force of such an incentive is considerable where students are technically ill-prepared, as Artigue notes in referring to "the help CAS can offer as assistants to computation and symbolic proofs for students with limited technical background" (§V.3).

This relates to the second contribution that Artigue identifies CAS as making to the development of mathematical knowledge: as a means of facilitating and extending experimentation with mathematical systems, including generalisation:

> [W]hat counts is the potential that CAS offers for obtaining results very quickly, for reconsidering a previous computation and substituting a parameter to a numerical value in it. (§V.3)

As Artigue notes, this second contribution is the more widely recognised one, notably in the North American work which has been discussed. But this paper has shown that a contribution of the first type –through the development of instrumented technique- was also present –if not fully acknowledged- in that work. Artigue is certainly right to describe the two contributions as complementary as long as extremes are avoided: one in which students struggle to use CAS productively, compounded by an absence of institutional norms for such use; another in which overly structured use of CAS in highly controlled situations denies students the opportunity to develop rich instrumentation schemes.

NOTE

[1] Because this paper was prepared before a paginated version of Artigue (2002) was available, references have been made to sections (§) and notes (#) rather than to pages.

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