Linking algebraic and geometric reasoning with dynamic geometry software

Final report to the Qualifications and Curriculum Authority

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1 Introduction to the study

This study was commissioned in October 2002 as one element of a 3-year QCA project on Algebra and Geometry at Key Stages 3 and 4, scheduled for completion in March 2003. The aim of the project –as a whole- is to provide advice and support for schools so they can develop and strengthen the place of algebra and geometry within the mathematics curriculum framework. The aim of this study –as a particular component of the project- is to investigate the potential of dynamic geometry software (DGS) to support curricular activities which link algebraic and geometric reasoning, in the light of new developments in the capabilities of such software.

1.1 The international context of the study

Previous work within the project (Hoyles, Foxman & Küchemann, 2002; Sutherland, 2002) has compared the treatment of geometry and algebra in the current mathematics curriculum for English schools with that in other educational systems.

1.1.1 Approaches to geometry

The geometry report (Hoyles, Foxman & Küchemann, 2002) found that although there were substantial differences between curricula in the conception and organisation of geometry, it was possible to identify a common core of content concerned with measures, angles and shapes. The report describes the English intended curriculum as emphasising experimental approaches in geometry through 'discovering' rather than 'proving' properties and relationships (p. 2). It suggests that the difference between the intentions of the English curriculum and those of countries such as France, Japan and Singapore is that the latter provide "clear[er] goals and progression for content in geometry – either as an object in its own right or as a tool for problem-solving or proof" (p. 3). The report indicates that "all the countries surveyed place a greater emphasis on geometrical constructions and on working in [three dimensions] than in England, although how these constructions are actually expected to be performed… varies between countries" (p. 3). Finally, while most systems see ICT as playing a part in the geometry curriculum, specifications of its role are usually marginal or fragmented. The exception is Singapore which provides "detailed guidelines for teachers on the integration of ICT into mathematics with dynamic geometry software and calculators commonly in use" (p. 3).

1.1.2 Approaches to algebra

The algebra report (Sutherland, 2002) found that the curriculum in England (in common with other anglophone systems) gives emphasis to "algebra as a means of expressing generality and patterns", in contrast to emphases elsewhere on "symbolising mathematical relationships", on "link[ing]… different representations of variables", or on "algebra as a study of systems of equations" (p. 2). It indicates that "the idea of introducing algebra within the context of problem situations" is widespread across systems, but that "these problem situations are sometimes more traditional word problems… and are sometimes more 'realistic modelling situations'" (p. 3). While the report comments that most systems "make an explicit reference to the use of new technologies in the curriculum" (p. 4), the infrequent mention of ICT in the remainder of the report, including the tabulated curriculum summaries, suggests that –as in geometry- specifications of its role tend to be marginal or fragmented.

1.1.3 Use of computer-based tools and resources

Internationally, an indication of the extent to which computer-based tools and resources are used in lower-secondary school mathematics is provided by the recent international surveys conducted as part of the original and repeated implementations of the Third International Mathematics and Science Study [TIMMS] (Beaton et al., 1996: 168; Mullis et al., 2000: 217-8). Averaged across the 23 educational systems participating in both surveys, the proportion of eighth-grade students reporting some degree of computer use in their mathematics classes rose, between 1995 and 1999, from 17 per cent to 21 per cent, while the proportion reporting such use as occurring more than 'once in a while' remained steady at 5 per cent. In the first survey, England –at 55 per cent- was the system with easily the largest proportion of students reporting computer use, followed by the United States –at 31 per cent, and both these figures were considerably higher than the median across systems –at 11 per cent. In the second survey, England was joined by Singapore, the system showing by far the most marked increase over the intervening period –rising by 44 percentage points. However, the median change across systems was a rise of 2 percentage points. Moreover, the proportion of students reporting using computers more than 'once in a while' was only 10 percent in England, 11 percent in the United States, and 5 per cent averaged across systems. Typically, then, computer use remains low, and its growth slow.

1.2 The national context of the study

In bringing observations from these international studies to bear on the current English curriculum for ages 11-16, fuller account needs to be taken of important aspects of the national context.

1.2.1 Pre- to post-16 progression in the mathematics curriculum

One important aspect of national context is progression from the compulsory study of mathematics (between ages 11 and 16) into optional mathematics courses (after age 16); in particular, to courses leading to AS- and A2-level awards in Mathematics, and to the free-standing mathematics units on <u>Working with algebraic and graphical techniques</u> and in <u>Modelling with calculus</u> which can contribute to an AS-level award in Use of Mathematics. The essential point is that these post-16 courses are largely built around a synthesis of algebraic and geometric ideas and methods to focus on important forms of quantitative covariation. They give no emphasis to geometry as a self-contained system, and relatively little to algebra in these terms. This indicates the importance of building a secure base in the pre-16 curriculum to anticipate this synthesis of algebraic and geometric technique. In particular, the international curricular comparison points to the importance of approaching algebra in a way which gives suitable emphasis –in their terms- to 'symbolising mathematical relationships' and 'linking different representations of variables', as well as to 'expressing generality and patterns'. It also suggests that current curricula may underestimate the potential of geometrical problems and investigations as a productive focus for such activity, alongside 'traditional word problems' –limited in scope and open to stereotyped presentation- and 'realistic modelling situations' –vulnerable to contextual ambiguities and noisy data.

1.2.2 Current national initiatives in secondary education

Another factor is the development of the *Framework for teaching mathematics* as part of the Key Stage 3 National Strategy. This document has elaborated the National Curriculum specification for Mathematics to provide much more detailed guidance on curriculum content and teaching approaches. The published rationale emphasises algebra as generalised arithmetic, but also acknowledges those aspects concerned with relationships between variables, functions and graphs

(DfEE, 2001: p. 14). This rationale emphasises geometry as “the study of points, lines and planes and the shapes that they can make, together with a study of plane transformations”, identifying as “a key aspect… the use and development of deductive reasoning in geometric contexts”, but also acknowledges linkage to “accurate drawing, construction and loci”, and to “work on measures and mensuration” (p. 16). Particular attention is drawn to the way in which “ICT offers good opportunities to develop geometrical reasoning and an appreciation of shape and space” (p. 16), with Logo and dynamic geometry software singled out for mention. Relationships between geometry and algebra are not emphasised, although mention is made of how “the use of formulae [in mensuration] can be linked to work in algebra, and can be enhanced by the use of spreadsheets and graphical calculators” (p. 17). In respect of linkage between algebra and geometry, then, the published rationale for the *Framework* does not fully anticipate the character of progression to post-16 study of mathematics. In addition, as will be explained more fully in due course, neither *Framework* nor *National Curriculum* acknowledge the distinctive expertise -and so learning- required to make effective use of computational tools for mathematical purposes.

1.2.3 Current use of computer-based tools and resources

A further important issue is the current situation of ICT use in secondary mathematics departments. Reporting on the general state of mathematics at secondary level, OfStEd (2002a: p. 2) judges that “use of ICT still requires improvement in many schools”, and indicates that the problems are not simply ones of access to equipment. Even in schools where resources are available, “only a minority of the mathematics teachers have the expertise and confidence to use them with pupils, despite recent training initiatives”; and, even with trained teachers, “work with computers is often restricted to occasional topics as part of the scheme of work in one or two year groups”. Moreover, OfStEd (2002b: p.9) suggests that use of ICT is rarely integrated into schemes of work and lesson plans: “Too often, teachers’ planning and schemes of work lack any reference to specific ICT applications, and pupils have difficulty recalling when they have used ICT in mathematics”.

Looking specifically at the effect on secondary mathematics of government initiatives promoting ICT in schools, OfStEd (2002c) reports that “most mathematics departments do not have ICT as a priority” with “recruiting difficulties, staff changes and the Key Stage 3 Strategy… generally seen as higher priorities” (§18). Consequently, the inspectors judge that “in only about one in three of departments is ICT mentioned in subject development plans, despite the current initiatives such as NOF-funded training and NGfL”, while “in nearly half, the leadership lacks a clear vision about how the use of ICT might develop and there are few strategies for monitoring and evaluating ICT work in the department” (§18). The report characterises the more established uses of ICT in the following terms: “graphic calculators are used to explore relationships between functions and graphs; graph-plotting software is used to link co-ordinates and shapes; low attainers are motivated in consolidating angle ideas [through drill games]; and spreadsheets are used for algebraic modelling” (§6). The inspectors’ observations of limited ICT use are corroborated by the findings of the Impact2 study (Harrison et al., 2002: p. 21) which found 67% of Key Stage 3 pupils and 82% of Key Stage 4 pupils reporting that they never or hardly ever used ICT in mathematics lessons.

Particularly significantly for this study, OfStEd (2002c: §32) reports that “very little use is currently made of the powerful dynamic geometry or algebra software available”, and this same phrase recurs in OfStEd’s most recent report (2002b: p. 9). It seems that advocacy of the use of such software in reports from joint working groups of the Royal Society/Joint Mathematical Council on the teaching and learning of algebra (RS/JMC, 1997) and of geometry (RS/JMC, 2001) has not been translated into support for curriculum revision and professional development.

1.3 Design of the study

1.3.1 Rationale for the study

These observations suggest that there is scope for increased attention in the English secondary mathematics curriculum to theorised analysis of mathematical situations following informal exploration of them; and to symbolisation of mathematical relationships as well as exploration of data patterns. This report will show that DGS have considerable potential to support such enhancement of the curriculum.

1.3.2 Available technical and didactical resources

Several DGS are currently available, but those most widely discussed in education are Cabri Geometry[1] and the Geometer's Sketchpad (GSP)[2]. Resources for the educational use of these DGS in teaching and learning geometry have been developed over the years; for Cabri, primarily in France, and for GSP, primarily in the United States, although there have been some pioneering British examples. A useful overview is provided by the Autumn 2002 issue of the journal *Micromath* (published by the Association of Teachers of Mathematics) which is devoted to such resources (including those developed by a working group of the association) and is accompanied by a CD-ROM of relevant material. Similar material can also be found in the manuals and demonstration files accompanying both packages, and on the websites of their publishers.

An important interest of this study is in the potential of a relatively recent enhancement of DGS which has added a graph plotting capability. While both the main DGS incorporate this capability to some degree, an upgraded version of Cabri Geometry is due for release which may match the additional facilities for graphing currently provided by Geometer's Sketchpad. Under these current circumstances, then, GSP will be used to exemplify DGS in this document.

An internet search was conducted with the aim of identifying material illuminating the potential contribution of DGS in teaching and learning algebra. Currently, the few relevant resources available are to be found primarily in the material accompanying GSP and on its website. These resources refer to standard graph plotting activities, notably investigations of the effects of changing parameters in function expressions. Other ideas include the use of algebra tiles to provide a visual model for algebraic factorisation, and the graphing of relationships between geometrical measures. Reference will be made to all of these in due course. Another source is the text on Teaching Mathematics with ICT by Oldknow & Taylor (2000: pp. 89-90) which has a short section on algebraic modelling with dynamic geometry, as does the recent publication on ICT and Mathematics produced for the Teacher Training Agency by a working group of the Mathematical Association chaired by Adrian Oldknow (TTA/MA, 2002: pp. 43-44).

1.3.3 Phase 1: Developing and organising didactical resources and concepts

Having searched widely (particularly in Britain, America, France and Singapore) for relevant resources, it became clear that the use of DGS in relation to algebra and its links to geometry was still in the early stages of development. Consequently, rather than being able to collect and analyse already developed resources in the first phase of the study, it became necessary to devise ways in

[1] An introductory description is available at http://www.chartwellyorke.com/cabri.html, a restricted demonstration version at http://www.cabri.com/en/downloads/, and further information at http://www-cabri.imag.fr/index-e.html.
[2] An introductory description is available at http://www.dynamicgeometry.co.uk/, a restricted demonstration version at http://www.keypress.com/sketchpad/sketchdemo.html, and further information at http://www.keypress.com/sketchpad/.

which existing ideas for use of DGS could be given an algebraic dimension, and algebraic ideas could be expressed through DGS tools. Hence this final report documents a range of mathematical ideas drawing primarily on my own work in progress (Ruthven, in preparation), which may prove to have educational value once they have been more fully developed didactically.

1.3.4 Phase 2: Exploring school conditions for trialling and refinement of material

The second phase of the study explored the possibilities of working with teachers and pupils to develop the first phase material more fully into viable and valuable teaching and learning activities. Once the interim report on the first phase was approved in mid January 2003, it was used as the basis for canvassing further participation by teachers, schools and local authorities known to have some GSP capacity. A number of contacts had been established by early February, but none of these felt able to undertake trialling within the tight timetable imposed by the requirement that work should be completed by early March 2003. Few schools had the software, and even fewer mathematics departments were making active use of it. Feedback from many contacts indicated a need to train staff in the use of GSP before undertaking work of this type. Departments also had commitments to an established scheme of work, and difficulties in gaining timetabled access to school computer rooms at relatively short notice. All these considerations reflect the realities of the current state of –and constraints on- use of ICT in secondary mathematics as portrayed by OfStEd.

These considerations also resonate with my past experience of development activities of this type. The best example is a project conducted for the National Council for Educational Technology between 1988 and 1991, concerned with integration of graphic calculators into advanced mathematics courses (where, analogously, there was no tradition and little experience of use at that time). This project commenced with a programme of staff training and advance planning over the course of a summer term, in anticipation of trialling over the ensuing school-year with specific project classes. This process of training, planning and trialling continued over the full two-year cycle of the advanced level course, providing the basis for subsequent production of a professional development pack (Ruthven, 1992). In my view, developing a viable model for the integration of dynamic geometry systems into the secondary mathematics curriculum requires extended activity of this type, in view of the significant changes it implies in schemes of work, and the considerable demands on teachers. I would encourage QCA and/or DfES and/or BECTA to consider supporting a further project of this type.

1.4 Presentation of material

Throughout this document –designed to read on printed page as well as computer screen- screensnaps have been used to convey a sense of the activities under discussion. Clearly, the static –and possibly monochrome- images cannot fully convey the sense of working with dynamic –and often colour-coded- images. A folder of the relevant GSP worksheets accompanies this report, so that the reader can gain this important sense[3].

[3] The folder of worksheet files needs to be unzipped before use. It may prove necessary to open the worksheets from within GSP (using the option on the File menu). An evaluation copy of GSP can be downloaded from http://www.keypress.com/sketchpad/sketchdemo.html.

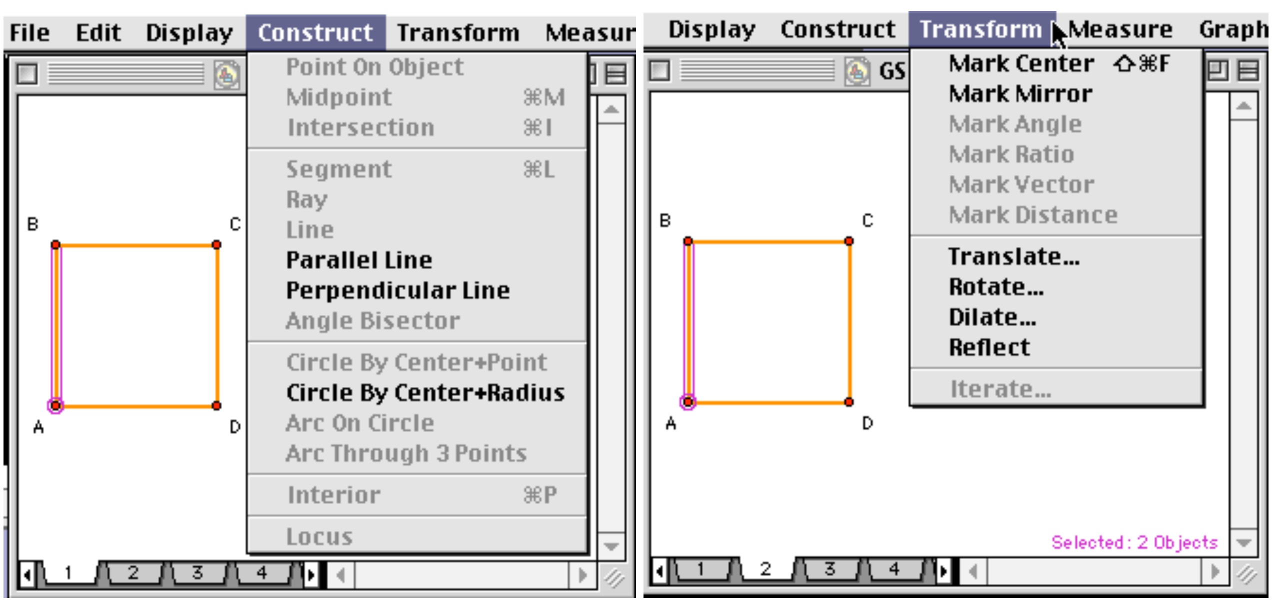
2 Working with dynamic geometry software

2.1 Basic features of the software

Dynamic geometry software is best known as a means of constructing and manipulating dynamic representations of geometrical objects in the plane. It provides tools supporting classical, transformational and coordinate methods of construction. Rather than creating a single static example of a generic geometrical object, the software makes it possible to create a dynamic construction which retains its defined characteristics but changes its visible form on the computer screen under manipulation. Such manipulation is applied to one or more defining elements by dragging with the computer mouse, directing with the keyboard arrows, or animating with the motion controller built into the software, allowing the displayed figure to range across cases. This can provide visual evidence -on the one hand- of any properties of the construction which are invariant across cases, and –on the other- of the locus of any variant element of the construction as another element is dragged.

2.1.1 Classical constructions

In GSP, basic tools for classical constructions can be selected from a Toolbox palette and used to construct points, circles or lines on the worksheet through direct use of the computer mouse. Constructions can also be made by preselecting defining elements already on the worksheet, and then applying an appropriate operation chosen from the Construct menu (Figure 2.1.1). This menu incorporates not only the recognised primary constructions with straight edge and compass, but standard secondary constructions -such as the midpoint of a preselected segment, or the line through a preselected point, running perpendicular or parallel to a preselected line[4].



Figures 2.1.1-2: The Construct and Transform menus of GSP.

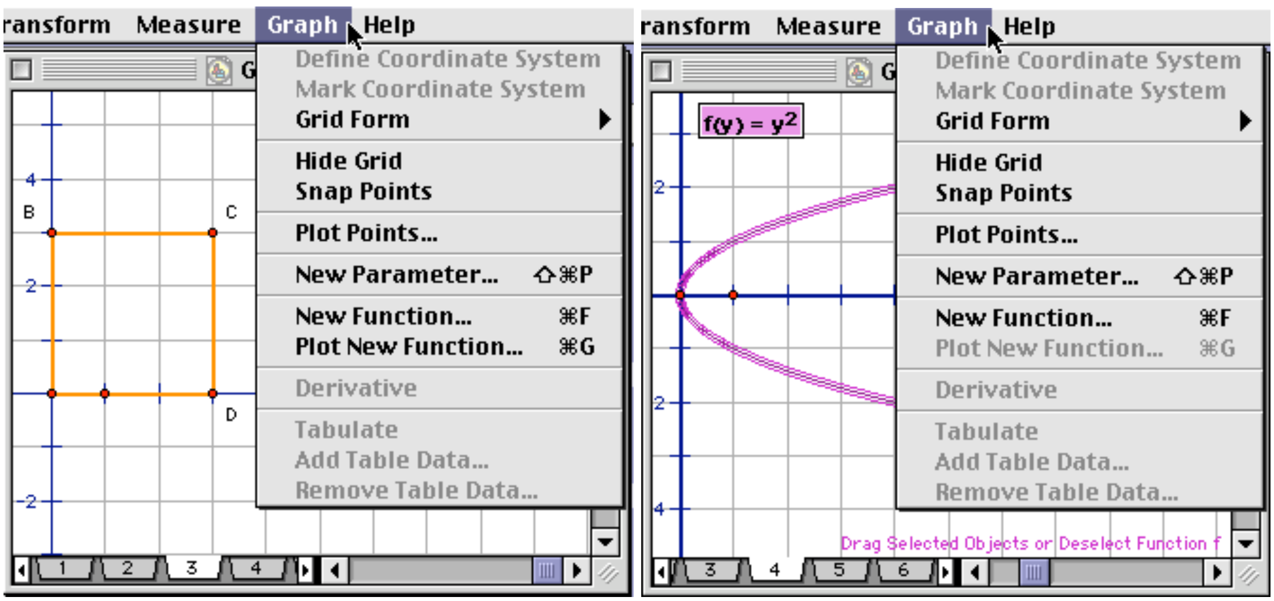
[4] In particular, all straight-edge-and-compass constructions specified in the *Framework* are available as direct operations. These secondary constructions can, of course, be reconstructed from the more limited set of primary ones, if desired in order to highlight underlying relationships.

2.1.2 Transformational constructions

Transformational constructions are carried out through applying the operations available on the Transform menu (Figure 2.1.2). First, preselected elements in the workbook are marked as the centre, mirror or other defining object for a transformation; then, the images of preselected elements are constructed in accordance with the chosen transformation. The standard transformations provided in the menu are translation, rotation, reflection and dilatation. The last of these permits independent specification of horizontal and vertical scale factors, in effect also providing for enlargement and stretch operations[5].

2.1.3 Coordinate constructions

The Graph menu (Figure 2.1.3) starts with operations which make it possible to define a coordinate system on the worksheet, and select its format. The grid may be square (identical scaling on horizontal and vertical axes), rectangular (distinct scaling on these axes), or polar. The positioning of the axes can be modified by dragging them or the point defining the origin. Likewise, the scaling of axes can be modified by dragging the point defining the unit, or other key values displayed on an axis. A grid to guide the eye can be displayed or hidden. A menu operation enables specified points to be plotted, permitting constructions based on coordinates.



Figures 2.1.3-4: The Graph menu of GSP.

2.1.4 Function graphing

The Graph menu also includes operations enabling functions (of the forms y=f(x), x=f(y), r=f(θ) and θ=f(r)) to be defined, plotted and edited (Figure 2.1.4). A definition can incorporate parameters, enabling a variable member of a class of functions to be defined. These parameters can be varied dynamically according to prior specifications, through use of screen sliders, arrow keys or animation commands.

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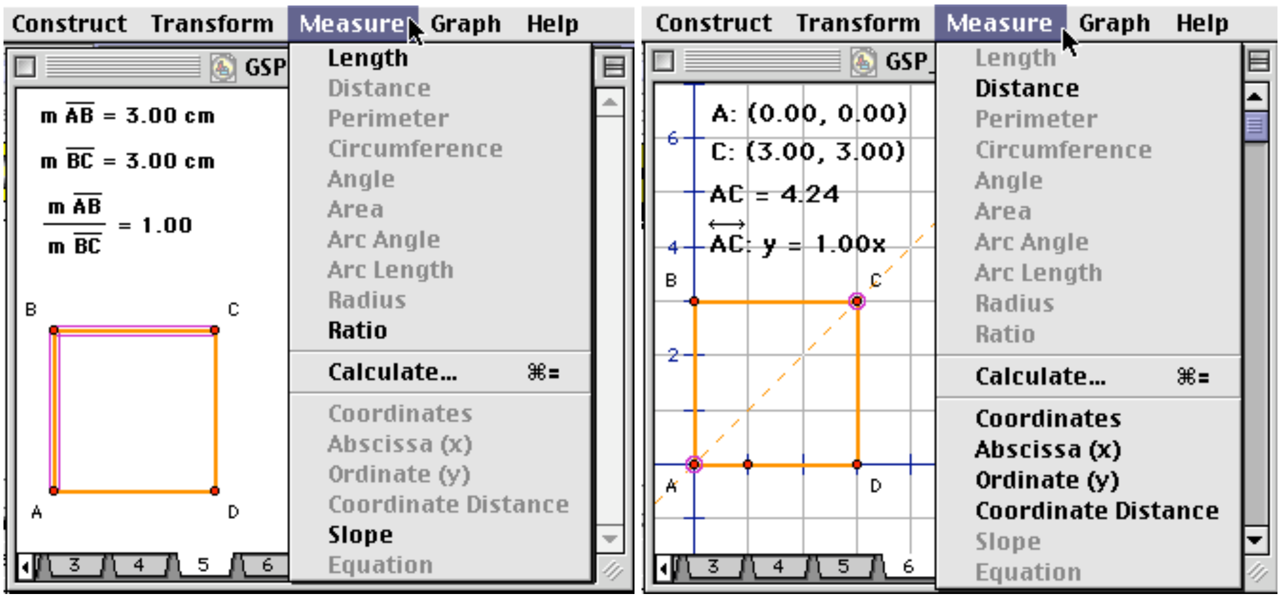
[5] Hence, all the transformations explicitly required by the *National Curriculum* are provided. In addition, the stretch transformation is available, which –although not explicitly required- plays an important part in formalising understanding of the scaling of coordinate axes.

2.1.5 Mensuration and calculation

The Measure menu (Figure 2.1.5) contains operations to measure elements of worksheet figures – notably lengths, areas and angles– and to carry out further calculations. Such measures and results can then be used in defining further elements and operations. The Measure menu (Figure 2.1.6) also contains operations to measure elements of worksheet figures in terms of their position relative to the coordinate axes. There are operations which return not just the coordinates of individual points and coordinate distances between points, but the coordinate equations of lines and circles.



Figures 2.1.5-6: The Measure menu of GSP.

2.1.6 Limitations of the software

As with other similar software, the positioning of points, lines and curves on a computer screen involves a degree of approximation, producing pixellated images to which the user must become accustomed, and which may be deceptive at times. In particular, visual evidence suggesting the coincidence or non-coincidence of elements cannot be trusted.

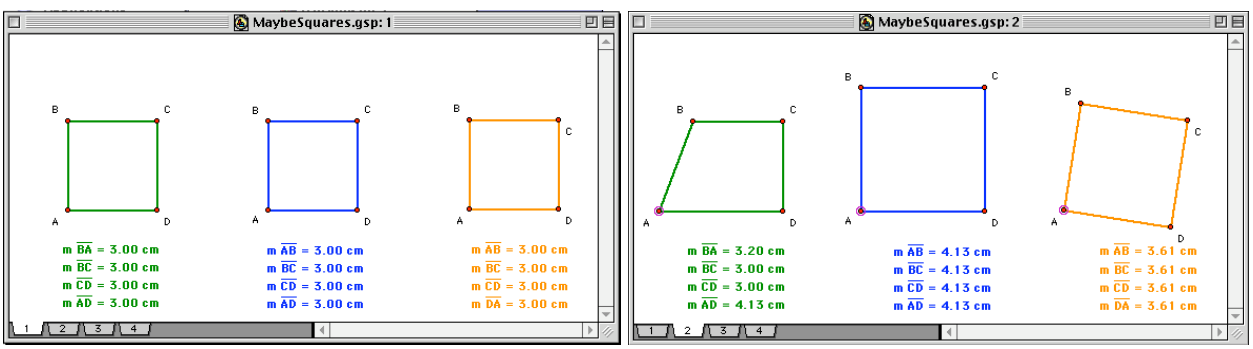
Equally, while it is possible to specify the degree of rounding with which measures are displayed, this rarely coincides exactly with the screen representation of objects. In particular, even when an element is dragged stepwise under the control of the arrow keys, measures are liable either to increase in steps larger than the smallest implied by the rounding specification so that key values are missed –e.g. 1.96, 1.98, 2.01, 2.03- or in smaller steps so that successive positions show the same measure –e.g. 2.0, 2.0, 2.0, 2.0. The consequence can be sufficient imprecision in measurement -and consequent manipulation- of objects to produce apparently incorrect results –e.g. 2.0 + 2.0 may be returned variously as 3.9, 4.0 or 4.1.

Finally, the terms in which coordinate equations can be expressed are restricted. The Equation operation in the Measure menu returns equations of (non-vertical) lines in a numeric $y = mx + c$ form (and of vertical lines in a numeric x = c form); and equations of circles in a numeric $(x - a)^2 + (y - b)^2 = r^2$ form. As noted earlier, the New Function command on the Graph menu permits the construction of functions in both y=f(x) and x=f(y) formats. For purposes of coordinate geometry, however, the important $ax + by + c = 0$ form is not available. This also limits the way in which simultaneous equations can be treated within the system.

2.2 Instrumenting mathematical activity

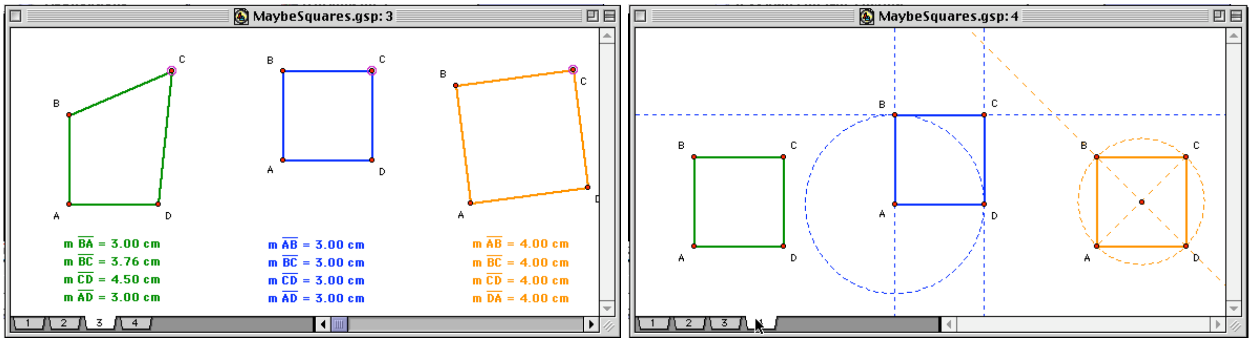
All mathematical activity depends on the use of tool systems. First, there are semiotic systems, such as those of classical or coordinate geometry, through which material drawings take on the character of mathematical figures. Then there are related technical artefacts, such as straight-edge and compasses, ruler, protractor and setsquare, or squared paper, which assist the construction and interpretation of material drawings as mathematical figures. Mathematical thinking is mediated by such tools. For example, the task of constructing a square is very different on squared paper from on blank paper. Equally, in the latter case, the task differs depending on whether straight edge and compasses, or ruler and set-square, are to be used. Moreover, there are important contrasts between the aspects of squareness associated with use of these different tool systems. Thus learning to use tools, and to coordinate the use of different tools for cognate purposes, has always formed an important component of a mathematical education.

It is important to consider the use of DGS in this light. This is true even where pupils and teachers use prepared templates which make no call for them to devise and produce constructions. To take the example of the square, Figure 2.2.1 presents a DGS screen on which three geometric figures – identical in appearance apart from their colour- are displayed. In a type of controlled experiment, the point A in each figure has been selected, and the arrow keys used to drag these points simultaneously in an identical manner, yielding the results illustrated by Figure 2.2.2. A similar controlled experiment has been carried out in relation to the points C as shown in Figure 2.2.3. This demonstrates that the properties of a mathematical configuration cannot be inferred from a single presentation of its DGS construction; what dragging reveals is that the configuration to the left is only coincidentally square in its opening presentation; and while dragging the other two configurations confirms the invariance of their squareness, that squareness has a different character, so that the same dragging action produces contrasting results.



Figures 2.2.1-2: Controlled dragging of apparently identical screen presentations reveals different underlying configurations.

Revealing the hidden construction lines for the configurations, shown in Figure 2.2.4, provides clues as to their differing behaviour. But reading and interpreting these clues involves much the same knowledge as devising the constructions themselves. DGS, then, provide no means of transcending the distinction between material drawing and mathematical figure; but they do constitute a powerful tool means for making that distinction –at the heart of many pupils' difficulties with geometrical argument– more explicit. In practice, then, any effective integration of DGS into mathematical activity in general (and that involved in the learning of mathematics in particular) must give sufficient attention to the ideas behind the construction of mathematical configurations to make their presentation as material drawings interpretable.



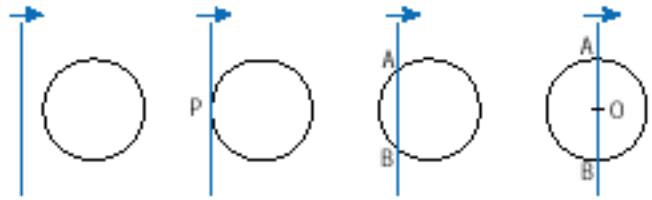
Figures 2.2.3-4: Further variation in the presentation of different configurations, and their construction lines revealed.

Here, it is important to recognise that DGS constructions are not identical to constructions with more conventional paper-and-pencil tools, just as constructions differ according to the conventional tool system used. In particular, then, a DGS is not something which can just be used by virtue of having mastered conventional tools of geometrical construction. There are, of course, many features in common; but there are also fundamental differences in operation which the expert user – to whom they have become transparent- is likely to gloss over. For example, there is an important difference in the degree to which the defining elements of mathematical objects are, and must be, made explicit. A line constructed in a DGS worksheet is explicitly defined by two points, whereas this is not necessarily so for a paper-and-pencil line drawn with a straight edge. Similarly, the various ways of constructing a circle within DGS all involve the centre being explicitly recognised as a point of the system. Likewise, to be used as the basis for investigation or construction, the intersection of two lines must be explicitly constructed in a DGS worksheet; and to draw a line parallel to another, a further point defining that new line must be made explicit. Anything beyond the most trivial construction, then, involves a higher degree of formal explicitness with a DGS than is necessary –and often customary– with pencil-and-paper. This introduces a degree of mathematical discipline into the use of a DGS, which is potentially educationally beneficial.

At the same time, however, this points to the need for a systematic development of pupils' capacities to 'instrumentalise' the DGS as a tool for mathematical activity, and to 'instrument' mathematical actions through use of the tool. These important issues have been most fully analysed in relation to the use of graph plotting and symbolic algebra tools (Artigue, 2002; Ruthven, 2002), and to some degree in relation to DGS (Laborde, 1993). They remain largely unrecognised in professional policy and practice regarding the integration of ICT into school mathematics. Taking the *Framework* as an example, the need to master conventional technical artefacts, such as ruler, protractor or compasses, have been recognised, and increasingly the distinctive qualities of using an arithmetic calculator. But although there is extensive reference to the use of graph plotting devices and dynamic geometry systems, there is not yet sufficient recognition of the importance of pupils developing the specific expertise needed to use these tools effectively: regarding, for example, the scaling of graphs in relation to graph plotting.

2.3 Mediating mathematical activity

Use **dynamic geometry software** to show a line and a circle moving towards each other.



Know that when the line:

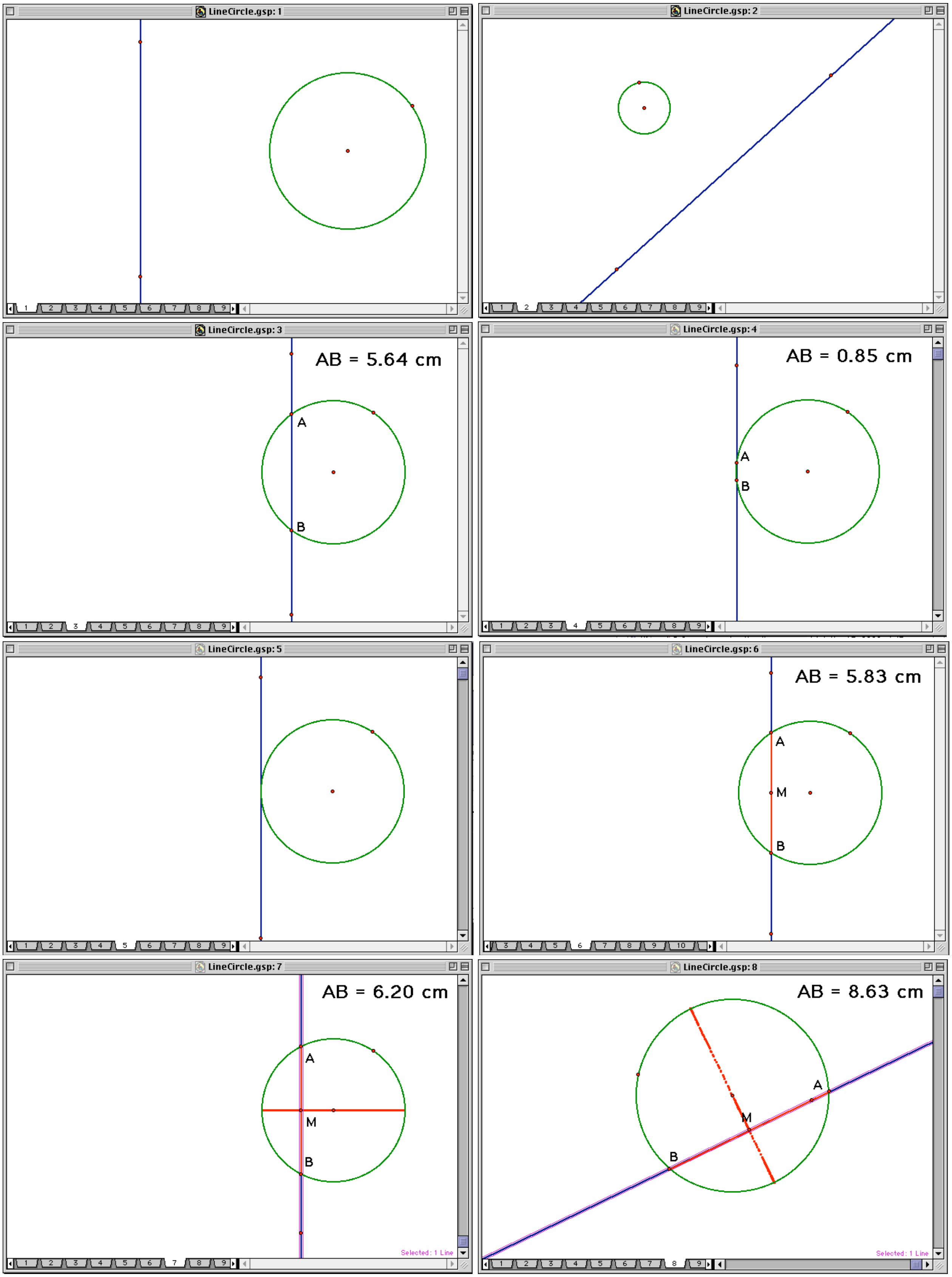
- touches the circle at a point P, it is called a tangent to the circle at that point;
- intersects the circle at two points A and B, the line segment AB is called a chord of the circle, which divides the area enclosed by the circle into two regions called segments;
- passes through the centre of the circle, the line segment AB becomes a diameter, which is twice the radius and divides the area enclosed by the circle into two semicircles.

Figure 2.3.1: Suggested use of dynamic geometry software in Framework (p. 195)

The task shown in Figure 2.3.1 is drawn from the *Framework* (p. 195), where it is placed at Y9 level. The important question to ask is what value using dynamic geometry software can add to this task which –other things being equal- could also be carried out by manipulating pointer or pencil in relation to a circle inscribed or placed on a classroom board or overhead projector. My argument is that a dynamic geometry system provides tools which assist the progressive appropriation of the initially informal actions and relations involved in the task to a disciplined framework of geometrical reasoning. As Laborde (1993: p. 53) argues:

> When asked to draw the tangent to a circle… students, by successive trials with a straight edge, draw a line touching the circle… but… the [mathematical] problem is not the production of the drawing of the tangent line but the determination of the point of tangency by means of geometrical relations.

Figures 2.3.2.1-10 show a sequence of screensnaps which trace the evolution of an investigation of the line-circle relation, leading into its theorisation. Figure 2.3.2.1 shows the initial construction of line and circle. Figure 2.3.2.2 emphasises the four degrees of freedom in this construction: provided by the two points defining the line, and the two points defining the circle, at centre and on circumference (all of which have been dragged). In pursuing an investigation, however, it is useful to be able to reduce the number of degrees of freedom. In this case, by regarding the circle as fixed (ruling out dragging it, or the points defining it, at least for the present), two degrees of freedom can be removed. Equally, by dragging the line as a whole (including both its defining points, rather than dragging one of these points individually) it moves parallel to its initial position, reducing the situation to be investigated to a single degree of freedom. Clarifying what is to regarded as fixed, at least for the present, and what is to be varied, and ascertaining how to achieve this with a dynamic figure, is a key step in systematising an investigation.



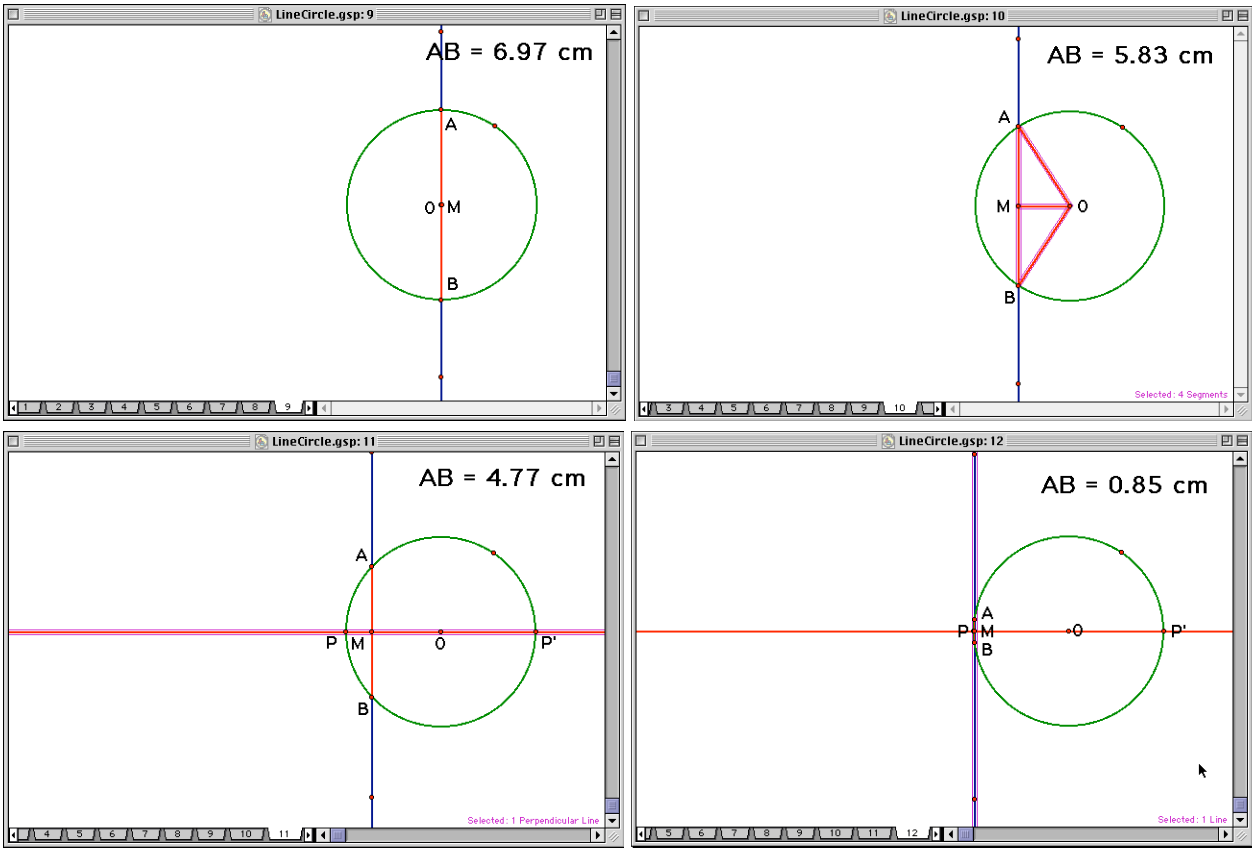
Figures 2.3.2.1-8: Investigation of the line-circle relation

Figure 2.3.2.3 shows a generic situation of the line cutting the circle. The points A and B have been constructed as the intersections between line and circle, then the segment joining them (with a display showing its length, to provide a helpful heuristic). As the line is dragged leftwards, A and B move closer (Figure 2.3.2.4) until both points (and the display of the distance between them) disappear (Figure 2.3.2.5). However, on the screen it is by no means clear that the line is no longer cutting the circle. Moreover, nothing displayed on screen during the course of the dragging establishes that there is a position where the line touches the circle at a single point, rather than cutting it at the two defined points. The viewer may make 'the touching hypothesis', but there is actually no visual evidence to support it.

To progress, then, it is necessary to go beyond the given construction. From considerations of symmetry, if there were to be a 'touching' point, then the 'average' of A and B would approach it as the line came closer and closer to the touching position. In Figure 2.3.2.6, this loose idea of 'average' is formalised in geometric terms as the midpoint M of segment AB, and the corresponding construction effected. In Figure 2.3.2.7, the line has been dragged from side to side, tracing the locus of M. This locus appears to be a straight line, and it appears to pass through the centre of the circle. But this figure might be some special case, or visually deceptive. To test these hypotheses empirically, the whole figure needs to be varied in terms of the four degrees of freedom identified at the start. Dragging the figure correspondingly (Figure 2.3.2.8) suggests that these apparent relationships are independent of the particular line and circle chosen for investigation.

To prove these hypotheses, however, it is necessary to theorise the situation further. Consider the particular position at which the line, and hence its segment AB, both pass through O (Figure 2.3.2.9). In this position of the line, O lies on the segment AB. Since OA and OB are both radii, O is equidistant from A and B. So O and M coincide. Hence, the locus of M does indeed pass through O. Consider next the more generic situation (Figure 2.3.2.10) in which the triangles OMA and OMB have been constructed. Again OA is equal in length to OB since both are radii. Likewise, MA is equal in length to MB, since M is the midpoint of AB. And finally, the triangles share the side OM. Hence, the two triangles are congruent. In particular, the angles OMA and OMB must be equal. And since these equal angles together form the straight angle AMB, they must both be right angles. This demonstrates that OM is perpendicular to the line defining A and B.

What this analysis establishes is a further property of M, that relates it directly to the two objects defining the figure –the moving line and the circle. M lies at the foot of the perpendicular from the centre of the circle to the moving line. In Figure 2.3.2.11, the extended perpendicular, and its intersections with the circle, P and P', have been constructed (and the segments used in the preceding analysis removed for greater clarity); as the line is dragged across the circle (holding it parallel –of course- to its original position), M moves along the perpendicular, between P and P'. Although, the visual evidence remains inadequate, it is now possible to argue theoretically, as follows, that the moving line touches the circle at the point P (and at the point P'). The line is shown approaching P in Figure 2.3.2.12. As the line is dragged through P, it is M which coincides with P (by virtue of the way in which they have been specified). In this position, since OP is a radius of the circle, so too is OM, as are OA and OB. Referring back to the triangles in Figure 2.3.2.10, this implies that the angles between OA or OB and OM are of size 0. Thus A and B also coincide with M when it takes position P. This is what it means for the line to touch the circle.



Figures 2.3.2.9-12: Theorisation of the line-circle relation

As the *Framework* recognises, such a full elaboration of the task as the one sketched here will not be accessible to all Y9 pupils: "More able pupils may be able to prove their conjectures analytically, but the formal use of congruent triangles is needed, and for most pupils this will be tackled in Key Stage 4" (p. 17). Nevertheless, this example illustrates how, in moving from investigation to theorisation, the mediation of a task by dynamic geometric software serves to introduce forms of geometrical discipline to the enquiry beyond those provided by the alternative means mentioned at the start of this section.

3 Visual models of algebraic relationships

Apart from those relating to use of its graphing facilities, the main algebraic materials accompanying GSP concern visual models of algebraic relationships.

3.1 Advanced algebra tiles

One of the standard templates accompanying GSP provides tools for using algebra tiles to model the products of two binomial expressions, through a concrete analogue of grid multiplication. The example shown in Figure 3.1.1 (reproduced from the corresponding proprietary GSP file) illustrates the use of (advanced) algebra tiles to model the identity between the product of the expressions 1 - 2x and x - 2 and the expression $2x^2 + 5x - 2$. Tiles -as shown in the key at the bottom right of the screen- are used to represent basic expressions. The sliders at the left of the screen allow the values of the variable(s) defining (the corresponding dimension of) these tiles to be varied dynamically. Because the width of the unit and x tiles is fixed at 1, the numeric values of their lengths and areas coincide; this allows them to 'represent' terms in either the binomial factor –by length- or the product –by area. The tiles for terms (including any preceding sign) which take a positive value for the current setting of the variable(s) are uniformly coloured, whereas those that take a negative value are diagonally quartered. Tiles are laid out to form the product as shown in the example in the top centre of the screen. The expressions to be multiplied are placed along the margins of the grid, and the product is modelled within the grid.

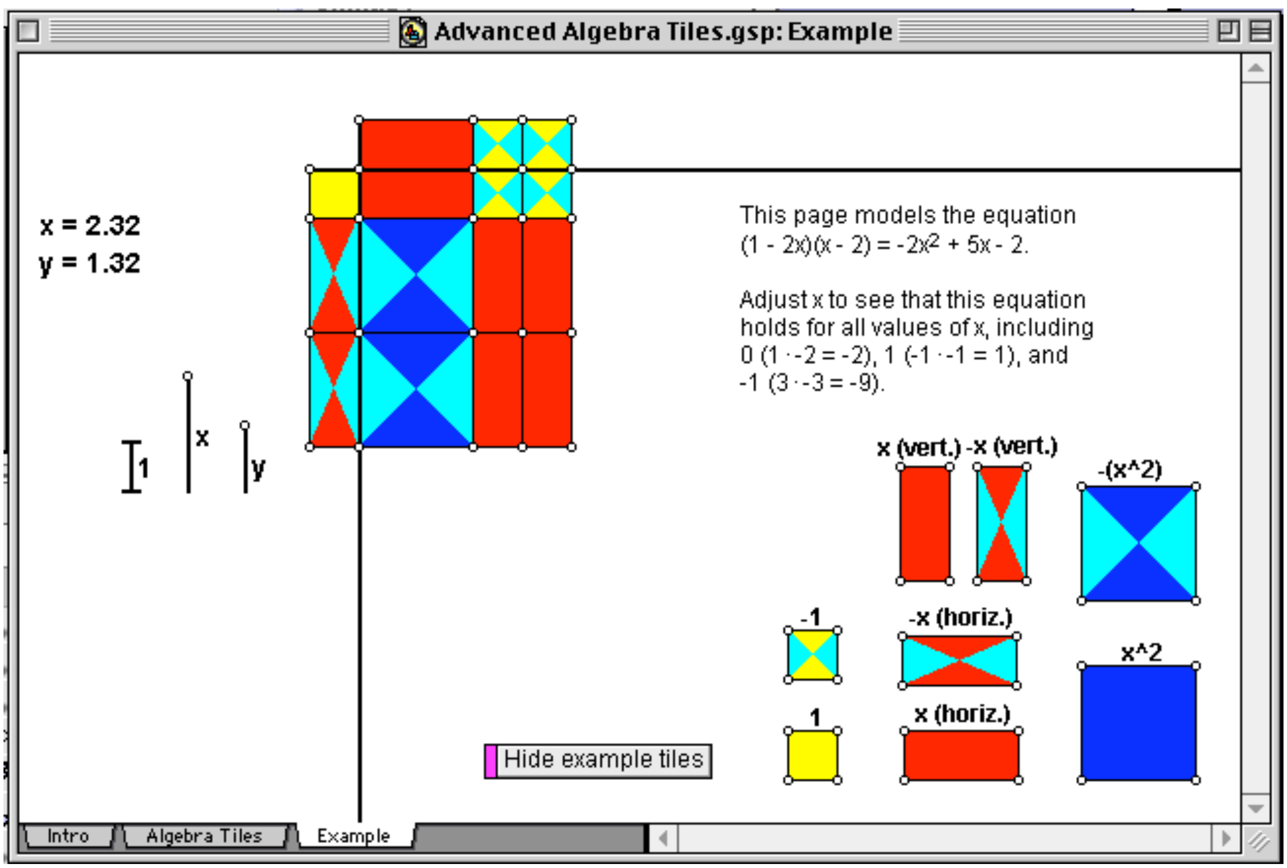


Figure 3.1.1: An algebra tile configuration modelling the product of 1 - 2*x* and *x* - 2.

Responsibility for the way in which the tools are employed resides with the user(s). Thus they can also be employed to construct unorthodox and erroneous models.

3.2 Trigonometric function generators

Another standard template accompanying GSP provides an animated illustration of the sense in which the trigonometric functions are also termed ‘circular’. Shown in Figure 3.2.1 (generated by using the corresponding proprietary GSP file), this animation demonstrates how the varying heights of two points (a quarter turn apart) on a rotating wheel generate sine and cosine graphs .

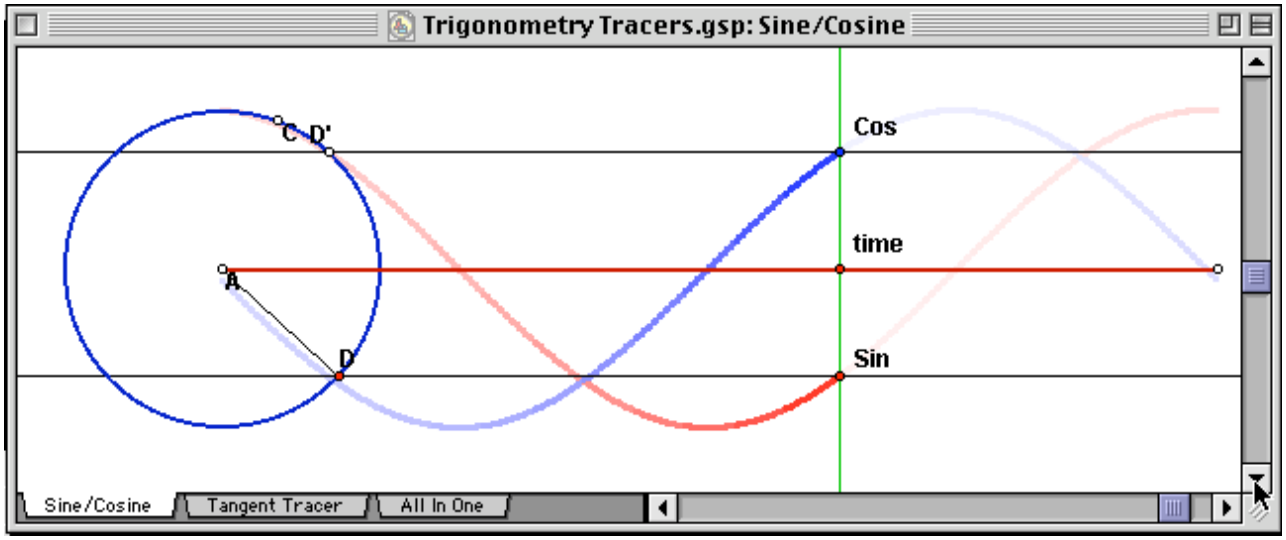


Figure 3.2.1: A trigonometric function generator.

3.3 Dynamic number lines

The algebraic idea of letter as variable can be represented in another way in GSP, through construction of a dynamic number line (Ruthven, forthcoming). This is a number line on which an independent variable and its dependent variables are plotted; as the point representing the independent variable is dragged along the line, the points representing the dependent variables move accordingly [6].

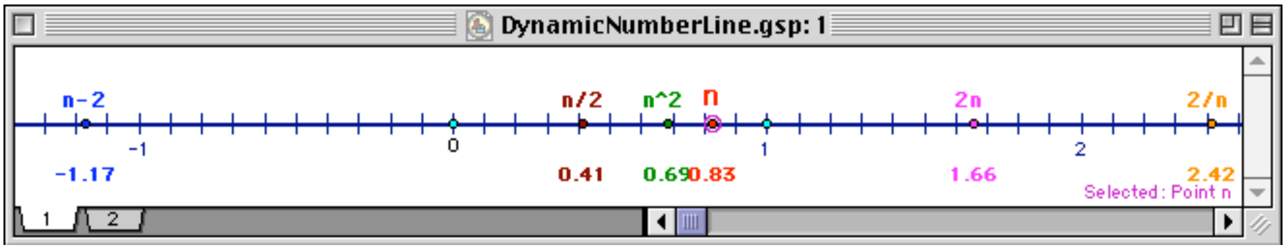


Figure 3.2.1: Screensnap of a dynamic number line showing an independent variable *n* and dependent variables *n*-2, 2*n*, n^2, *n*/2 and 2/*n*.

In the example shown in Figure 3.2.1, the independent variable is *n*, and the dependent variables are *n*-2, 2*n*, n^2, *n*/2 and 2/*n* respectively[7]. (For those reading this in monochrome, the points are colour-coded as well as labelled on the computer screen, to assist discrimination between them). As presented here, the value of a variable is shown in two ways –as a position on the number line, and as an associated decimal number[8]). As the independent variable *n* is dragged along the number line, the dependent variables move correspondingly. From the starting position shown, as *n* increases – moves to the right[9]- all other variables do likewise with the exception of 2/*n* which decreases – moves to the left. Likewise, as *n* decreases, so initially do all the other variables except 2/*n*. However, as *n* passes through 0, the direction of movement of n^2 reverses; it seems to 'bounce off' the 0 position; in addition, n^2 'overtakes' *n* as they pass together through 1. Similarly, as *n*/2, *n*, and 2*n* pass through 0, their relative positions reverse; and these three variables move along the number line at different, but clearly related, 'speeds'. By contrast, *n*-2 never exchanges position with *n*; rather, both move at identical 'speed', and maintain a fixed 'distance'. Equally, important insights

[6] In GSP, a dynamic number line can be constructed from the x-axis of a co-ordinate system (with grid and y-axis hidden). The variable 'driving' the model –in this case *n*- is constructed as a point (free to be dragged) on the axis, with its label edited to display the appropriate variable name. Other variables are then defined by calculation from n, and their positions on the number line plotted. The number line can be recentred and rescaled by the usual GSP means of dragging the line and its markers.

[7] This example is a more open version of a task proposed for Y9 students in the *Framework for Teaching Mathematics* (p. 9) under the head Applying mathematics and solving problems.

[8] There may be good practical and didactical reasons for hiding the decimal displays showing the values of variables. Practically, the problems of representing a continuous variable in the discrete medium of a pixellated screen are well known, and not peculiar to this situation; in particular, depending on the number –high or low- of significant figures chosen for the decimal display of a variable, the display may either 'skip' values –notably important values marked on the number line- or may 'stick' for some time at the same value –even though the point is moving on the number line. Didactically, hiding these values requires students to attend to the spatial positioning of points on the number line –as the only source of information about the variables. Equally, reading or estimating values using the visual scale is important in instrumentalising the number line. Initially, however, the provision of information about variables in both numeric and graphic form is likely to be important in helping students establish a basic concept of variable.

[9] To the expert it may seem redundant to write 'as as *n* increases –moves to the right-' but I do this to emphasise that one of the things that students are learning is precisely this extension to variables of the mathematical conventions surrounding the number line.

can be gained by focusing on instantaneous relationships between the values of variables; for example, n and $2/n$ coincide just as n^2 passes through 2 (as shown in Figure 3.2.2).

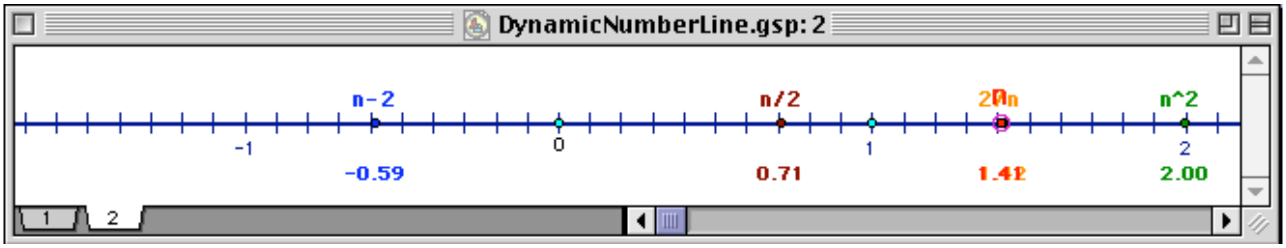


Figure 3.2.2: Screensnap of a dynamic number line showing an instantaneous relationship between the values of n, $2/n$ and n^2.

This dynamic representation of changing variables can serve, then, to highlight important mathematical phenomena calling for investigation and explanation, and to activate important supporting metaphors. And the ideas involved are precisely those underpinning fundamental issues including the solution of inequations such as $n^2 > n$, the non-solvability of equations such as $n = n-2$, and the relative rates of change of independent and dependent variables.

Nevertheless, why introduce dynamic number lines to represent variables when it would be both more practically convenient and more mathematically conventional to employ dynamic coordinate graphs? The key advantage of the dynamic number line is that it facilitates comparison of independent variable and dependent variable, because these are represented on the same line (so that positions and movements are directly comparable)[10]; this provides a quite different –and more primitive- sense of the covariation of independent and dependent variable than that available on a dynamic coordinate graph. Semiotically, the dynamic coordinate graph provides a more complex representation, with the independent variable scaled on one number line, dependent variable(s) on another orthogonal number line, and their values represented by the position of a point on a third line or curve forming the graph. The key advantage of the dynamic coordinate graph is that it holds information about a range of joint values of independent and dependent variables, not just their instantaneous state. Hence, the dynamic number line is more than just a simple precursor to the dynamic coordinate graph[11]; each representation facilitates attention to particular aspects of covariation.

[10] The earlier example –following the approach of the source task- incorporated an unusually large number of dependent variables: n-2, $2n$, n^2, $n/2$ and $2/n$. In practice, of course, it might prove advisable to work with rather fewer dependent variables on a single number line.

[11] And, of course, the transition from dynamic number line to dynamic coordinate graph can be assisted by initially showing values of the independent and dependent variables not only on the graph, but on their respective axes.

4 Covariation of measures in geometrical figures

This section illustrates the possibilities of analysing the covariation of measures in geometrical figures, including expressing and analysing such covariation in algebraic terms.

4.1 Varying the radius of a circle bounded by hexagons

This example illustrates the basic idea of co-varying measures. A circle is constructed, and then two regular hexagons; an 'inner' hexagon with vertices on the circle, and an 'outer' hexagon with edges touching the circle (as shown in Figure 4.1.1). Similar constructions are, of course, possible with other polygons. Variation –in the form of a simple scaling operation- is introduced by dragging the point defining the radius of the circle. This illustrates the invariance of the visual nesting of 'inner' hexagon within circle within 'outer' hexagon. This nesting is also reflected in the numeric ordering of measures of perimeter/circumference (as shown in Figure 4.1.1) and area (as shown in Figure 4.1.2). Even where pupils, have already grasped these relationships, this simple situation can serve to illustrate the idea of co-variation of measures.

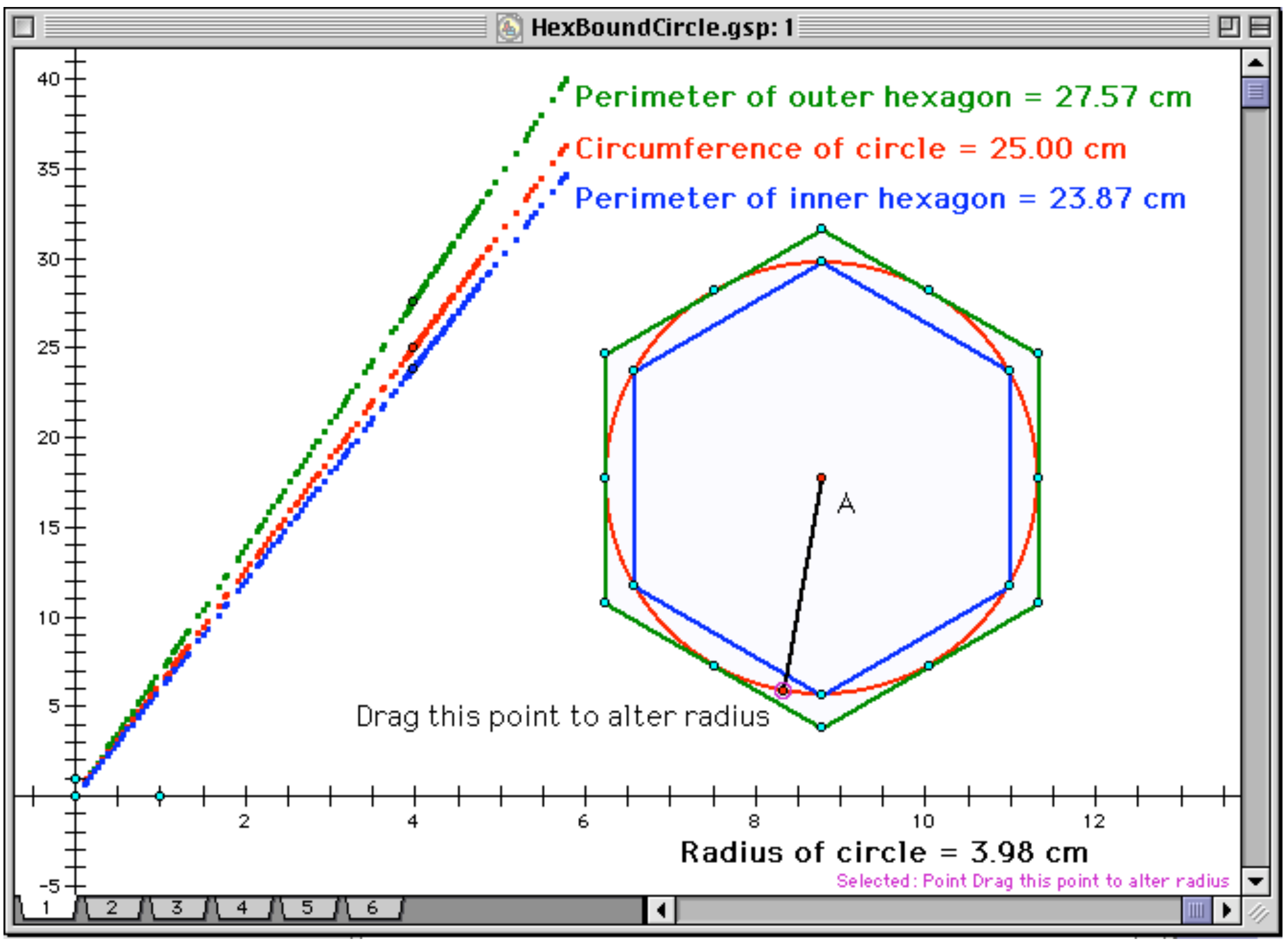


Figure 4.1.1: Investigating covariation of perimeter/circumference with radius measures.

The covariation of any two measures can be shown by plotting them as a co-ordinate pair. Dragging the radius in the geometrical figure results in the plotted point moving accordingly. This movement can be traced, showing how 'inner' perimeter, circumference and 'outer' perimeter co-vary with radius (Figure 4.1.1), as likewise do the three area measures (Figure 4.1.2).

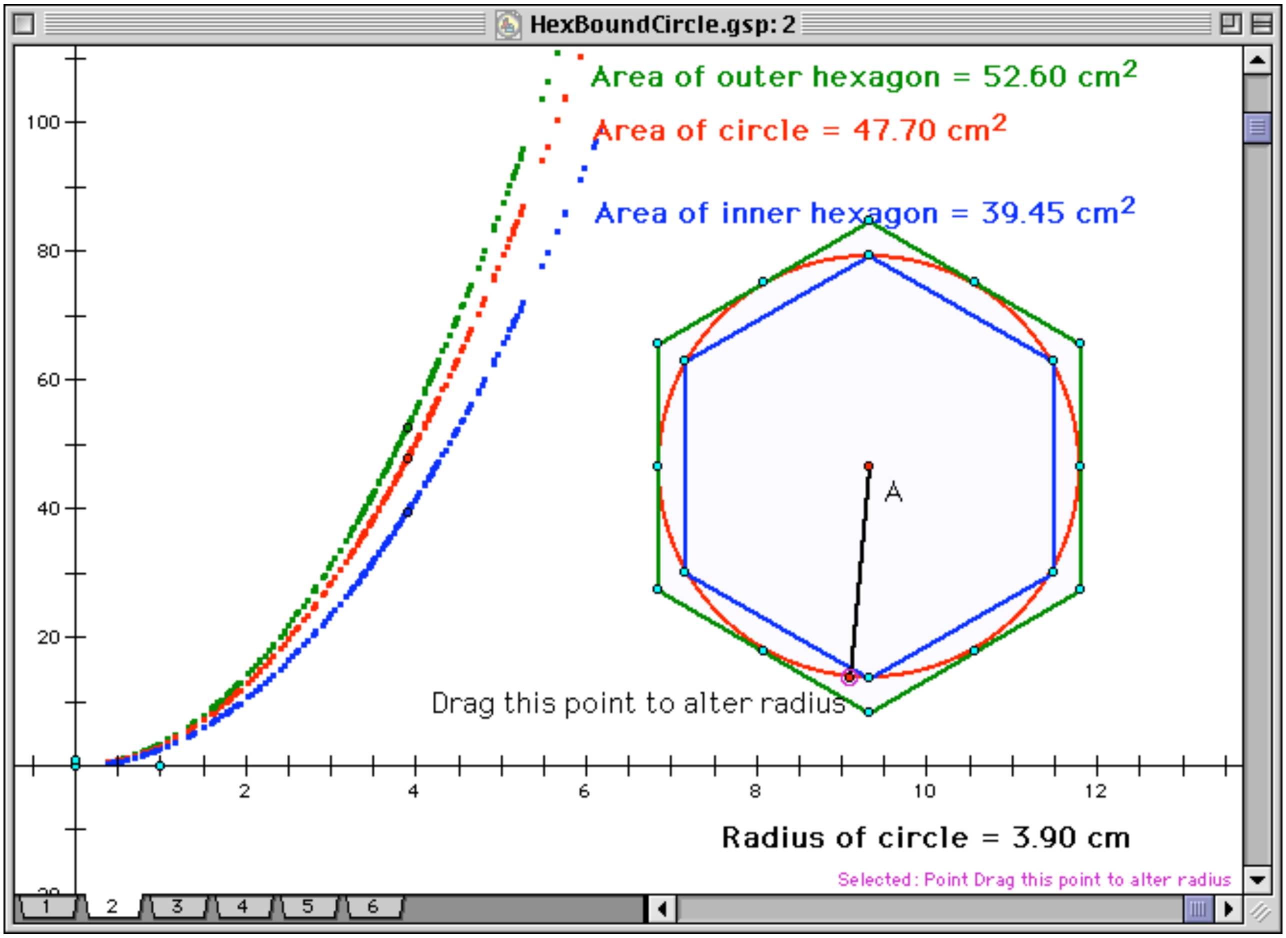


Figure 4.1.2: Investigating covariation of area with radius measures.

The most intuitively available form of covariation is between an independent measure –in this case the radius measure- and dependent measures –in this case the circumference/perimeter and area measures. There is, of course, another less immediately evident form of co-variation open to investigation and analysis: between the different dependent measures.

The traces may themselves suggest algebraic relationships between the variables. These can be tested by superimposing a plot of the conjectured relationship. Equally, given the requisite geometrical knowledge (notably of the properties of set-square triangles), formulae can be deduced for the perimeters and areas of 'inner' and 'outer' hexagons and squares in terms of the radius of the circle.

4.2 Covariation in a folded sheet of paper

This example illustrates how a dynamic geometry figure can support various forms of geometric, algebraic and trigonometric reasoning.

The basic construction shows the outline of a sheet of paper folded over so that one of its vertices touches the opposite edge, with the paper flattened to form a crease. In the basic construction (Figure 4.2.6), this vertex P can be dragged so that the endpoints of the crease remain within the boundaries defined by the edges AB and BC. The dimensions of the sheet of paper can also be altered by dragging C and D, introducing further generality to the situation.

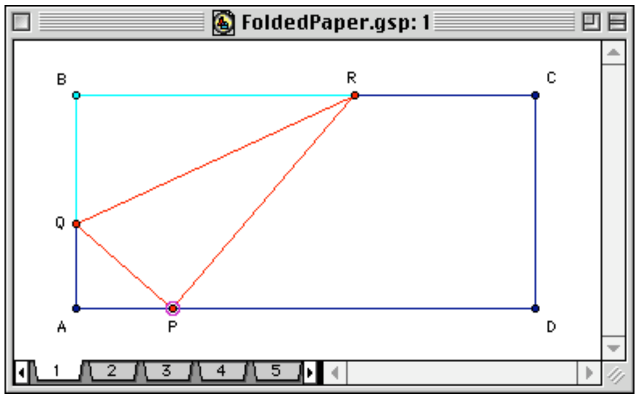


Figure 4.2.1: The basic folded paper construction.

A first geometric task is to establish the constraints on the movement of P on AB. Essentially, this involves identifying and analysing the extreme positions available for P on AB. In practice, the software may not be capable of displaying the extreme positions themselves, providing an incentive to analyse the situation geometrically (as shown in Figures 4.2.2-3).

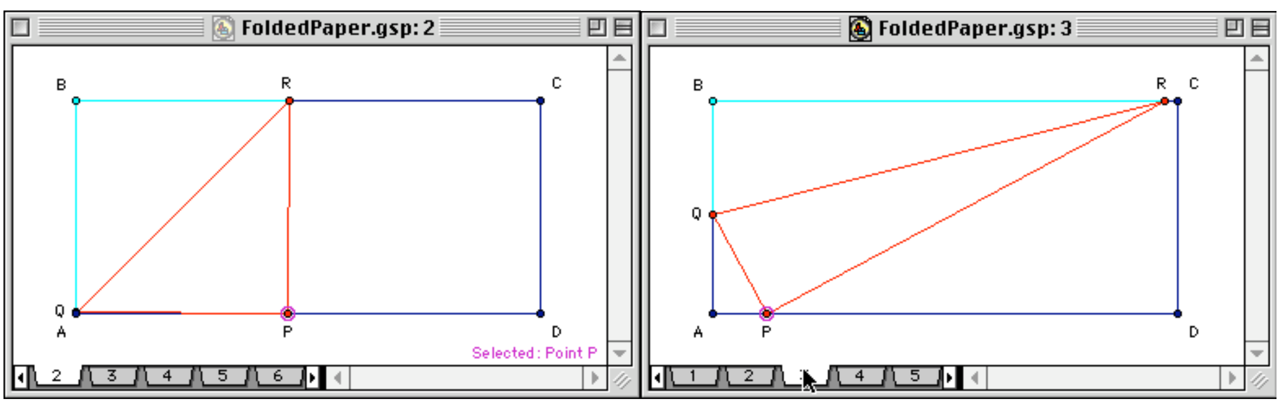


Figure 4.2.2-3: Approaching the extreme positions of the vertex P.

A second task focuses on angle relationships. With the measures of two angles displayed (as in Figure 4.4.4), the task is to identify and analyse any consistent relationship between them as P is dragged. This depends on using standard angle properties (notably those of complementary, supplementary and vertically opposite angles, and angles in a triangle) and can be developed into a systematic relationship expressed in algebraic terms. An extended task is then to conduct a complete analysis of angle relationships in the figure, so determining the minimum number of angle measures needed to deduce all others, and thus the minimum number of letters needed to express all angle measure variables. A similar task focuses on identifying systematic relationships between lengths within the figure and expressing them algebraically (with supporting mensuration as shown in Figure 4.2.5).

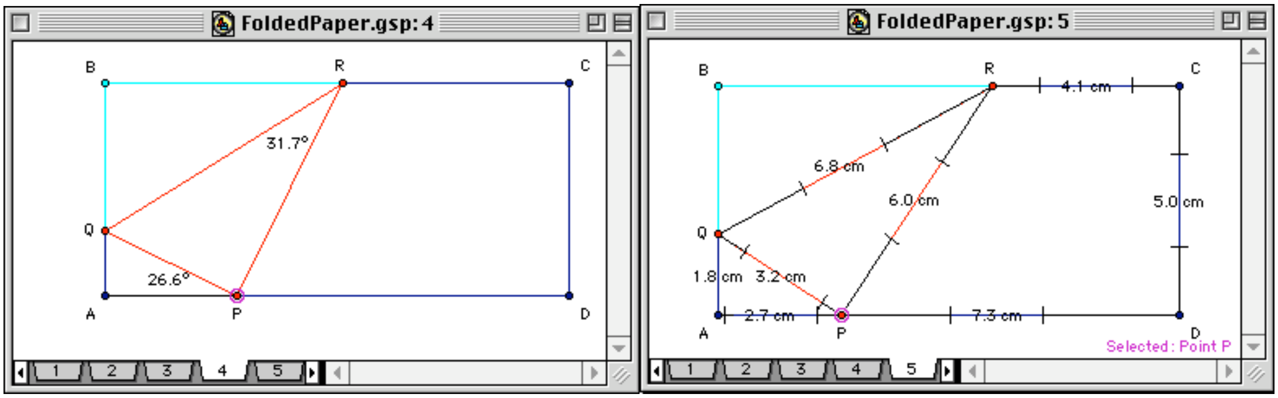


Figure 4.2.4-5: Basic displays for exploration and analysis of angle and length relationships .

Finally, for pupils familiar with the trigonometry of right-angled triangles, a fuller analysis relating angle and length measures is possible. One approach is to first pose the task of deducing as many other measures as possible from the information shown for the specific case in Figure 4.2.6. The methods adopted can then inform the development of generally applicable formulae which can be tested in a range of cases produced by dragging not just P, but C and D.

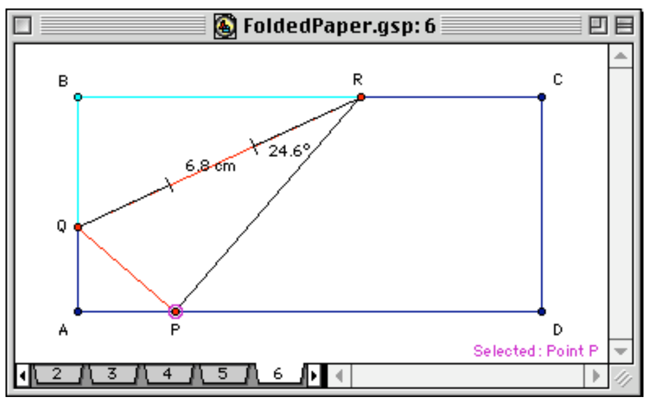


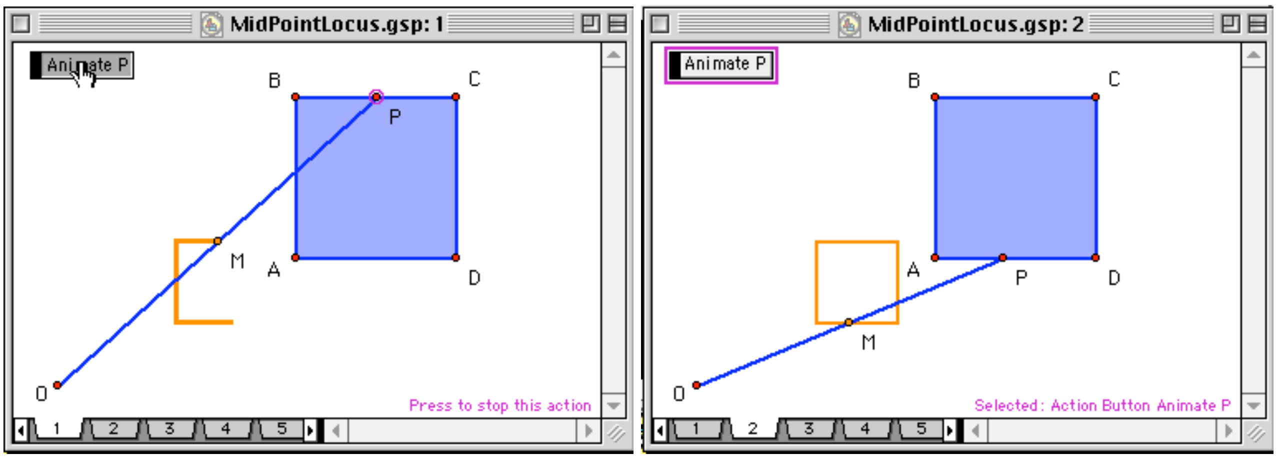
Figure 4.2.6: Basic display for full analysis of measure relationships.

5 Classical, transformational and co-ordinate approaches to locus

This section illustrates the possibilities of developing –and comparing- parallel analyses of locus problems in classical, transformational and co-ordinate terms –with the last linked to algebraic methods.

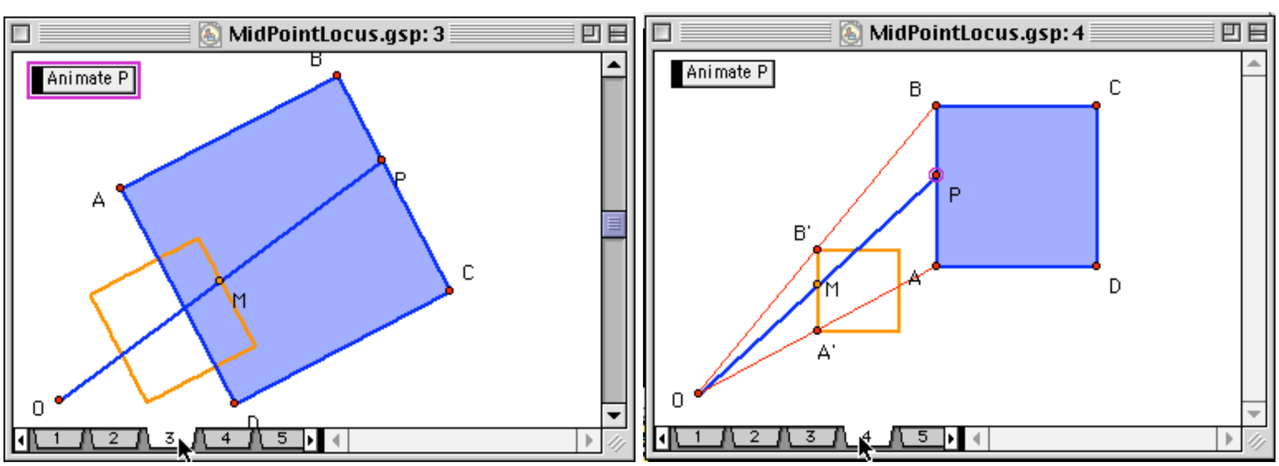
5.1 The midpoint of a constrained position vector

The type of problem situation considered in this subsection concerns the locus formed by midpoint of a line segment connecting a fixed point (in effect, an origin) to a point moving round the perimeter of a shape. Here, the example of a square is used, but any polygon would serve, as would a circle (while offering variation in some features of the ensuing analysis).



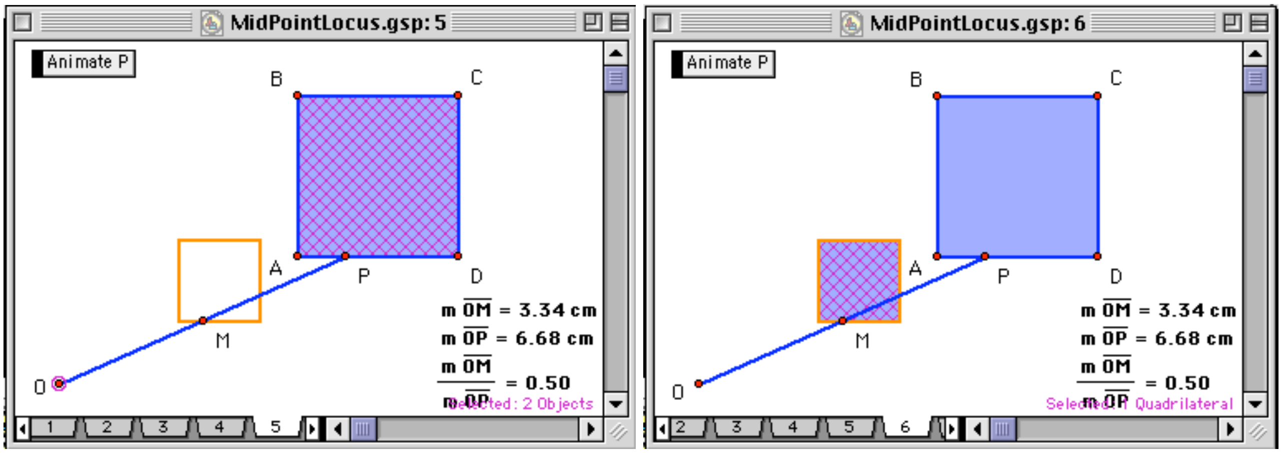
Figures 5.1-2: Locus formed by the midpoint, M, of a line segment connecting a fixed point, O, to a point, P, moving round the perimeter of a square.

As the point P moves (as animated in Figures 5.1-2), so does the midpoint M (as shown by its trace) in what appears to be a smaller version of the square. Dragging the defining objects –the point O and the square ABCD- indicates that the locus consistently appears to take such a form (as shown in Figure 5.1.3). The form can be analysed through classical geometrical methods by considering any edge of the square, and constructing the triangles formed by the line segment in its 'variable' position OP, and its 'extreme' positions OA and OB (as shown in Figure 5.1.4). The straightness –and half-size- of A'MB' can be proved through analysis of similar triangles.



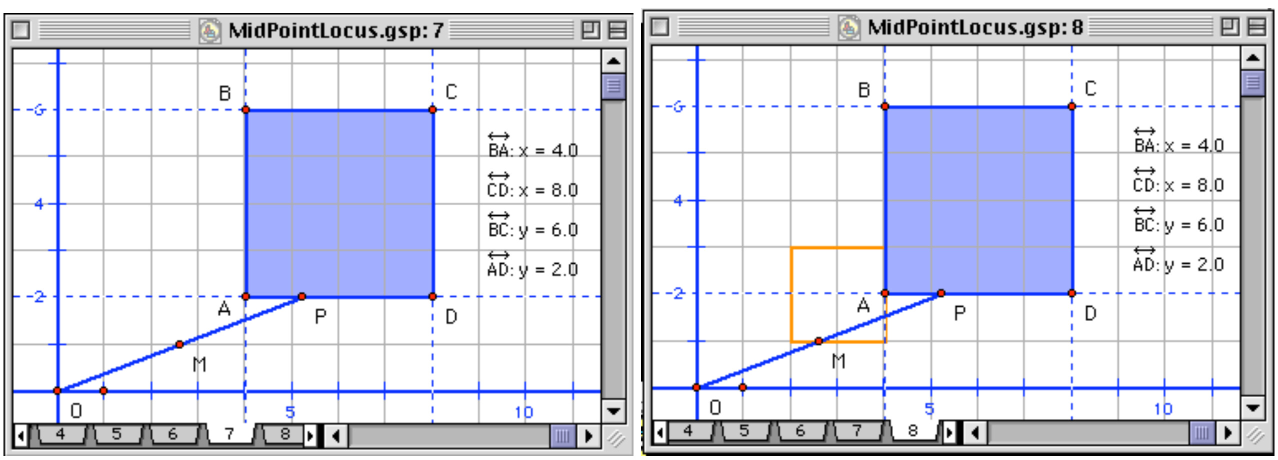
Figures 5.1.3-4: Dragging defining elements of the locus, and analysing it in classical terms.

In this light (and rather more directly) the relationship between the square on which P is moving and the shape formed by M can be conceived in transformational terms as a dilatation centred on O, with scale factor identical to the (given) ratio between the lengths of OP and OM (as shown in Figures 5.1.5 –before- and 5.1.6 -after).



Figures 5.1.5-6: Analysing the locus in transformational terms as a dilatation of the square.

Finally, the problem can be treated in co-ordinate terms. The fixed point is placed at the origin, and each of the edges of the square satisfies a co-ordinate equation for the corresponding straight line. The square may be positioned so that these equations are simpler in form (as shown in Figures 5.1.7 and_ 5.1.8) or more complex (as shown in Figure 5.1.9). In either case, the relationship between the equations of edges in the original square and its image can be examined.



Figures 5.1.7-8: Investigating the locus in co-ordinate terms.

Such an investigation points to a co-ordinate analysis of the relation. Although this should –at least initially- be conducted in relation to specific examples, it can be developed into a general analysis of the following form:

Take O as (0, 0), P as (p, q), and M as (m, n). The values of p, q, m and n vary as the points move. However, whatever the position of the points, the following can be said.

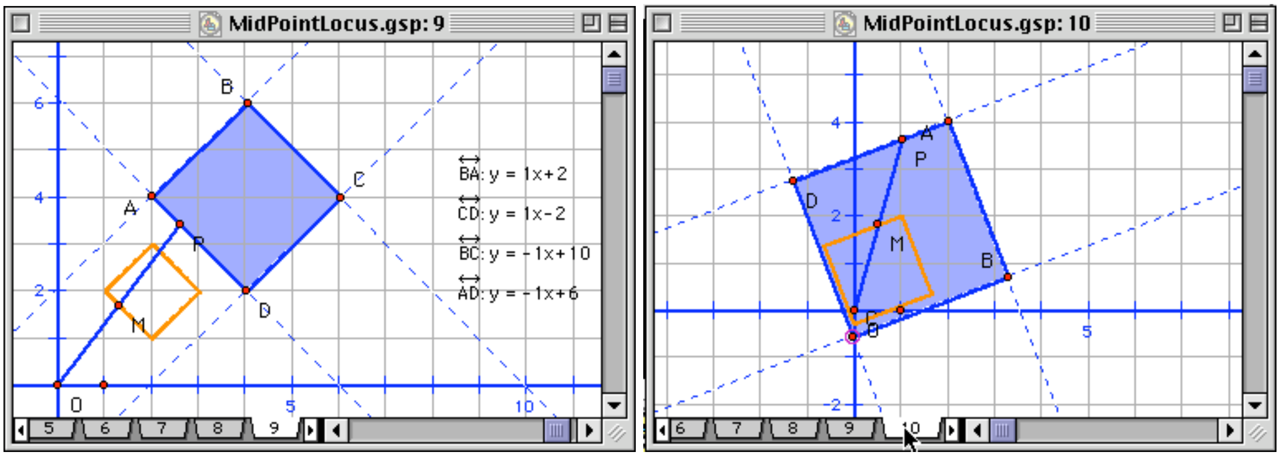
Since M is the midpoint of OP, $m = p/2$ and $q = n/2$, which is equivalent to $p = 2m$ and $q = 2n$.

If P runs along an edge satisfying equation $ax + by + c = 0$, then $ap + bq + c = 0$.

Replacing p and q in that result by their equivalents, $2am + 2bn + c = 0$.

Consequently, $am + bn + c/2 = 0$.

Hence whatever its position, the coordinates of M satisfy the equation $ax + by + c/2 = 0$.
This is the equation of a line parallel to the original edge on which P is moving, and half its distance from the origin.



Figures 5.1.9-10: Further investigating the locus in co-ordinate terms.

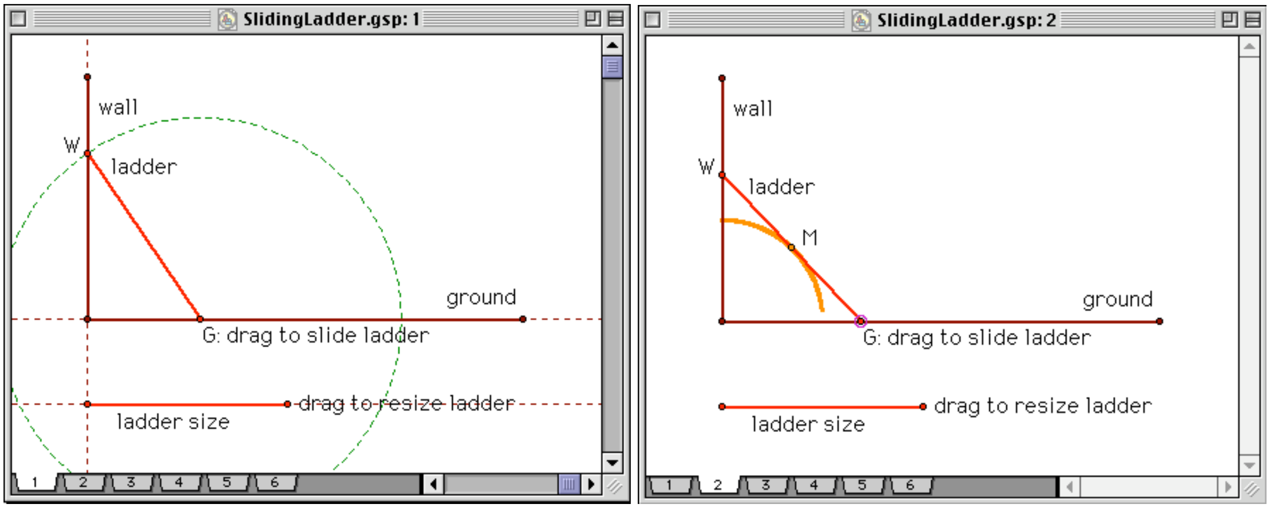
Further investigations give additional possibilities for analysis. For example, under what circumstances does the locus of the midpoint fall entirely within the original square (as in Figure 5.1.10), or completely outside (as in Figure 5.1.9).

This problem also lends itself to treatment in vector (co-ordinate pair) terms, using a parameter to express the position of P between the vertices defining an edge of the square.

5.2 The midpoint of a sliding ladder

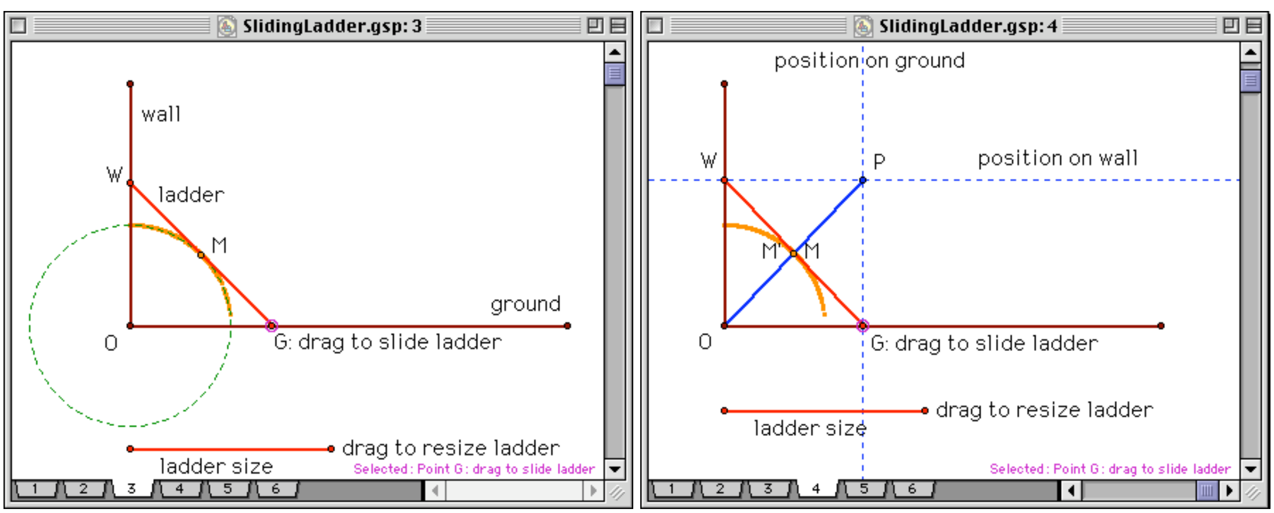
This subsection considers another midpoint problem: that of the locus of the middle rung of a ladder as one end slides down a wall and the other along the ground. Here, then, the segment concerned is of fixed length, and is constrained to move in a different way.

A basic construction for the problem is shown in Figures 5.2.1-2. Dragging G drives the sliding motion of the ladder; the path of the midpoint M is traced accordingly.



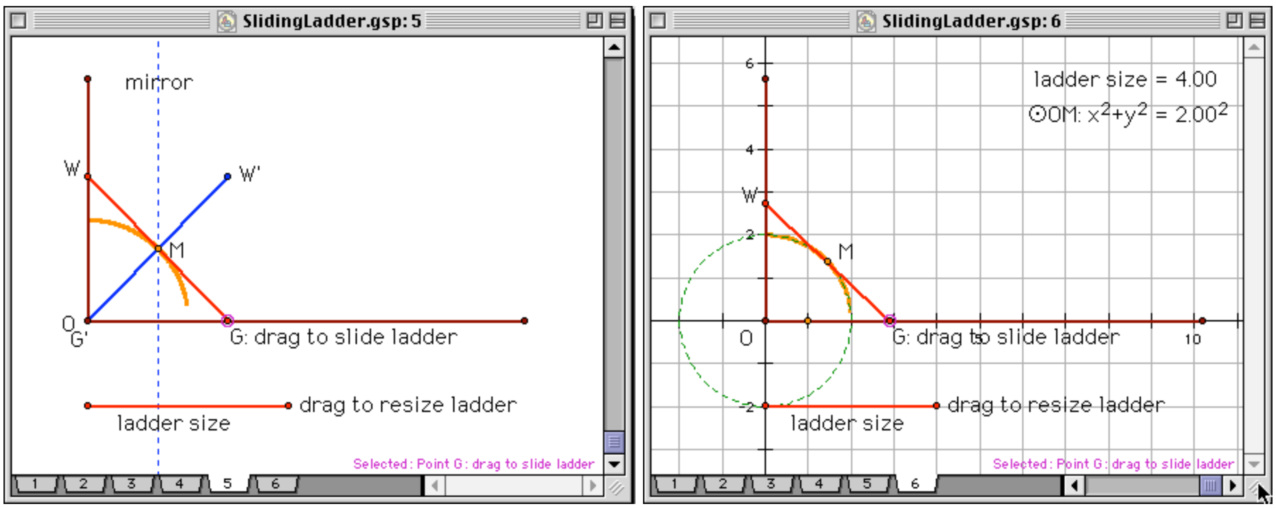
Figures 5.2.1-2: Basic construction for investigating the sliding ladder problem

It appears that the locus of M is a quarter-circle, centred at O. By constructing such a curve, its fit to the locus can be tested visually (as shown in Figure 5.2.3). But, this remains to be proved.



Figures 5.2.3-4: Testing the apparent form of the locus, and creating a classical analysis.

A classical analysis can be developed by constructing the (variable) point (P in Figure 5.2.4) defined as having the same height above the ground as that end of the ladder resting against the wall and the same distance from the wall as that other end resting on the ground. This point is constructed as the intersection of the corresponding lines parallel to ground and wall. Assuming that the face of the wall is perpendicular to the level ground (suggesting possible variants to be investigated later), this creates a (variable) rectangle (OWPG in Figure 5.2.4), in which (by a standard property of rectangles) the diagonal OP is of the same (fixed) length as the diagonal GW (which is the ladder), and these diagonals intersects at their midpoints M' and M. Consequently, the locus of M is identical to the locus of M'; and since O is a fixed point, and OM' of fixed length, M' ranges over a quarter circle lying between the ground and the wall. In the case of any residual uncertainty, this analysis can be tested by dragging G and observing whether M' and M always coincide. By doing so, another way of viewing the problem is suggested. Not only do the two diagonals rise and fall together, as if they were pivoted at their midpoints, but the trajectory of the line OP resembles that of another identical ladder falling away from a starting position up against the wall.



Figures 5.2.5-6: Creating transformational and co-ordinate analyses of the locus.

This suggests an alternative transformational analysis, based on reflecting the ladder in the (variable) mirror line running parallel to the wall and passing through the midpoint of the ladder (as shown in Figure 5.2.5 by the reflection of GW to create its image G'W', both having midpoint M). Here, in order to establish that the locus corresponds to the quarter-circle centred on O, it is necessary to establish that, whatever the position of the sliding ladder GW, G' always coincides with O.

Finally, the locus can be analysed in co-ordinate terms (as shown in Figure 5.2.6, where O has been placed at the origin, G on the x-axis, and W on the y-axis). Again, specific numerical examples are likely to be more accessible for pupils, but some will be able to build up to a fully general analysis, as follows:

Take O as (0, 0), G as (g, 0), W as (0, w), and the size of the ladder as s. The values of g and w vary as the ladder slips, but s remains constant.

However, whatever the position of the sliding ladder, the following can be said.

Since M is the midpoint of GW, it has co-ordinates ($g/2$, $w/2$).

Since GOW is a right-angled triangle, $g^2 + w^2 = s^2$.

Consequently $g^2/2^2 + w^2/2^2 = s^2/2^2$, which is equivalent to $(g/2)^2 + (w/2)^2 = (s/2)^2$.

Hence, whatever its position, the co-ordinates of M satisfy the equation $x^2 + y^2 = (s/2)^2$.

This is the equation of a circle, centred at the origin, with radius half the size of the ladder.

6 Geometrical properties of algebraic graphs

Access to graph plotting software has encouraged investigative teaching approaches to the study of families of algebraic graphs –notably linear and quadratic- based on exploring the effects on the graphs of varying the values of parameters within their defining equations. Although vivid, such effects can easily be geometrically underinterpreted or misinterpreted by pupils in the absence of any control or check. For example, the graphic effect of varying the parameter m in the linear equation $y = mx$ can be underinterpreted simply in terms of the changing y-coordinate of the point $(1, m)$, or misinterpreted as a rotation. A key potential of DGS is their capacity to introduce greater geometrical discipline and insight into such investigations.

6.1 Properties of linear graphs

One way of encouraging such discipline and insight is to construct geometric tools to support the analysis of function graphs. Figure 6.1.1 shows a tool for use in investigating linear graphs.

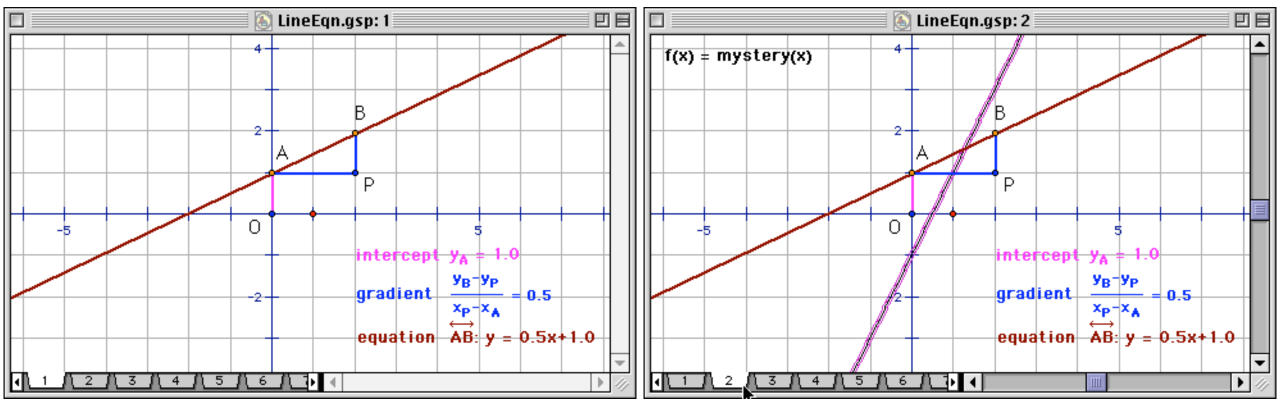
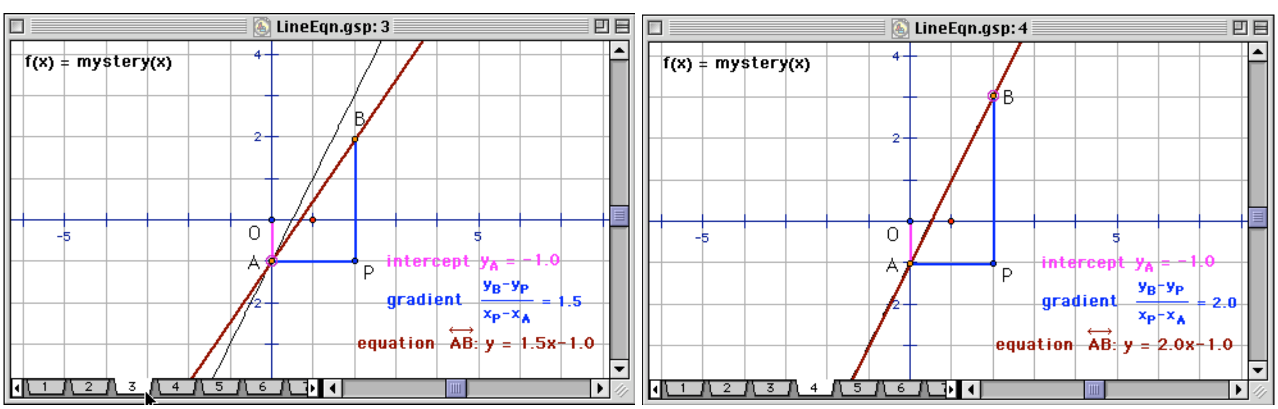


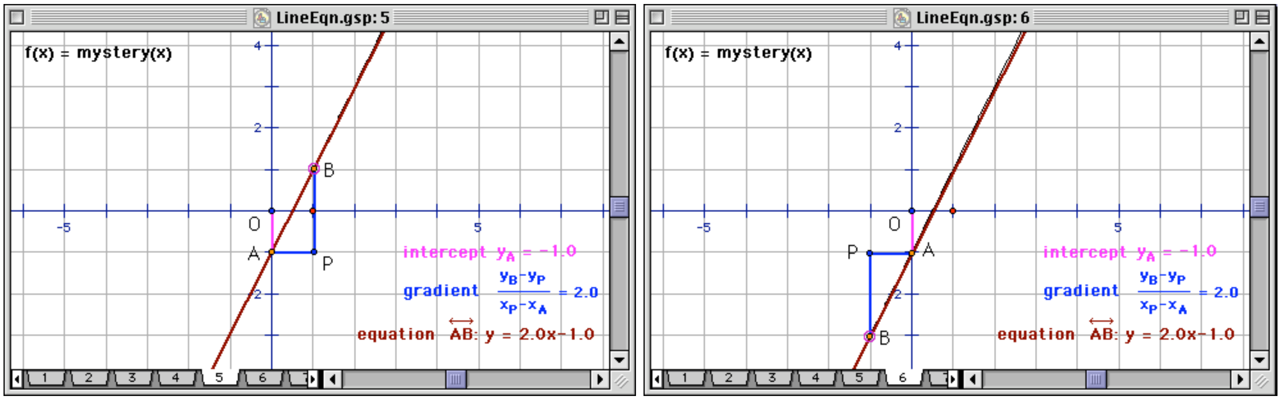
Figure 6.1.1-2: A simple tool for investigating linear graphs.

The tool consists of a line to be manipulated so as to lie along the graph under investigation. Associated with this line is a segment joining the origin O to the point A (defined as the intersection of the line with the vertical axis); this serves to highlight the geometrical basis of intercept. Equally, the point B can be dragged freely, defining a pair of segments PB and AP which indicate the vertical and horizontal components of any section of the line; this serves to highlight the geometrical basis of gradient. The application of this tool to the graph of a mystery linear function is shown in Figure 6.1.2; by dragging the points A and B (Figures 6.1.3-4) the tool is positioned over the graph, identifying intercept, gradient and equation.



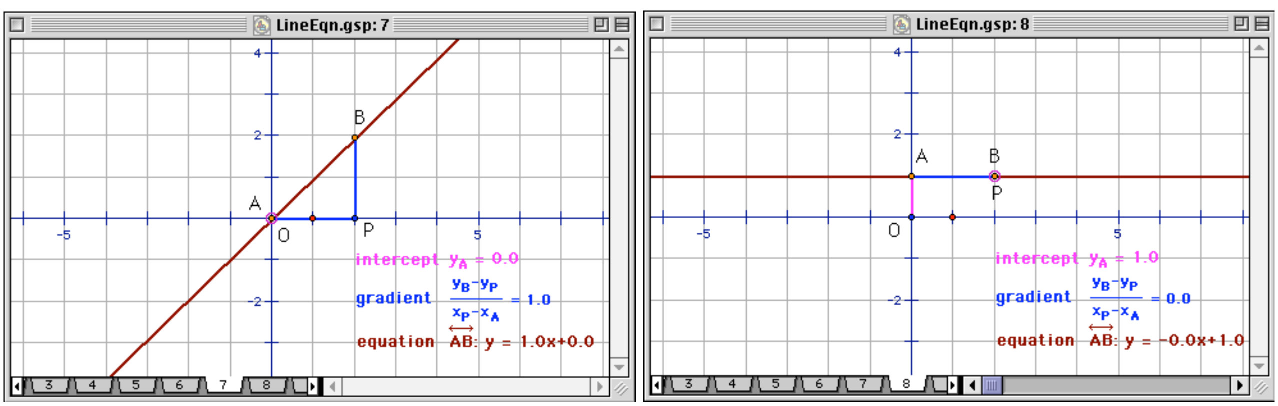
Figures 6.1.3-4: Applying the tool to the graph of a mystery linear function.

Exploiting the freedom of the point B by dragging it along the function graph emphasises gradient as the invariant ratio of vertical to horizontal component (as shown in Figures 6.1.5-6).



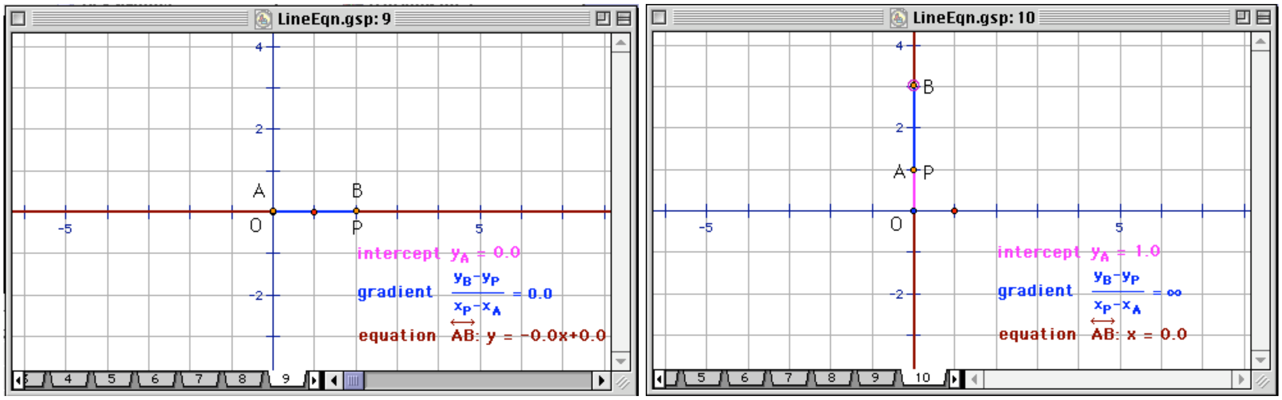
Figures 6.1.5-6: Freedom of movement of the gradient component of the tool.

In effect, this tool embodies the intercept/gradient model of linear graphs, so that mastering the manipulation of its components involves developing schemes corresponding to this mode of formal mathematical analysis. Such schemes cover situations in which the intercept indicator appears to vanish and the constant term in the equation becomes 0.0 (Figure 6.1.7) -corresponding to an intercept at the origin with y-coordinate 0; or the gradient indicator collapses and the coefficient of the x term in the equation becomes 0.0 (Figure 6.1.8), corresponding to a horizontal graph of constant value with gradient 0. Equally, in a situation where the equation of the line displays an explicit zero coefficient, this can be compared with the counterpart functions defined by equations in which the corresponding term is either included or excised.



Figures 6.1.7-8: Special configurations of the tool signal zero intercept or zero gradient.

Finally, manipulating the tool to lie along one or other of the axes explores their characterisation as graphs of linear functions. The horizontal axis can, of course, be characterised in this way (Figure 6.1.9), whereas the vertical axis cannot (Figure 6.1.10) –dragging A reveals that this is the unique line on which every point could be considered an intercept!

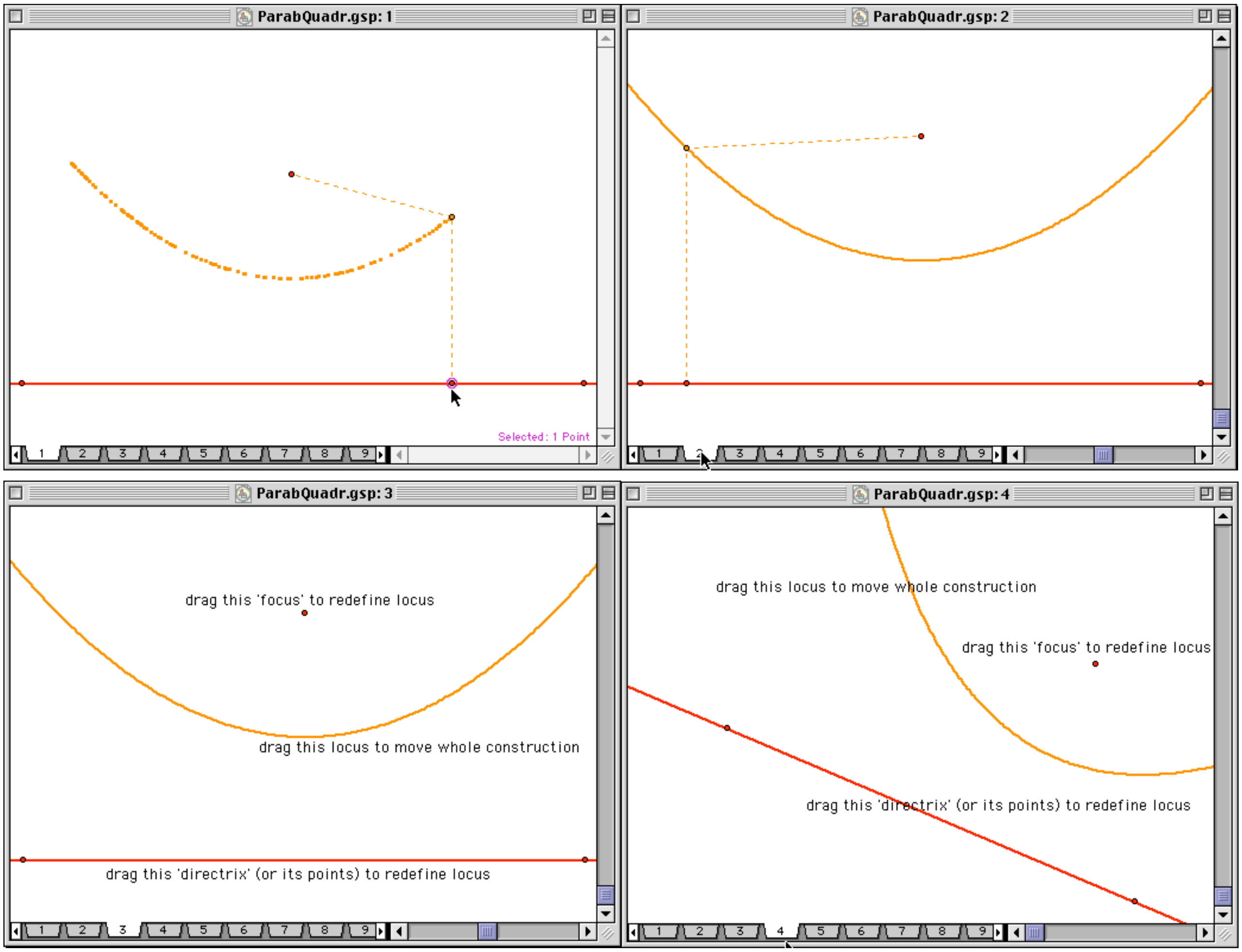


Figures 6.1.9-10: Attempted characterisation of the axes as linear function graphs.

6.2 Properties of parabolas and quadratic graphs

This section focuses on the use of a parabola tool in analysing quadratic graphs.

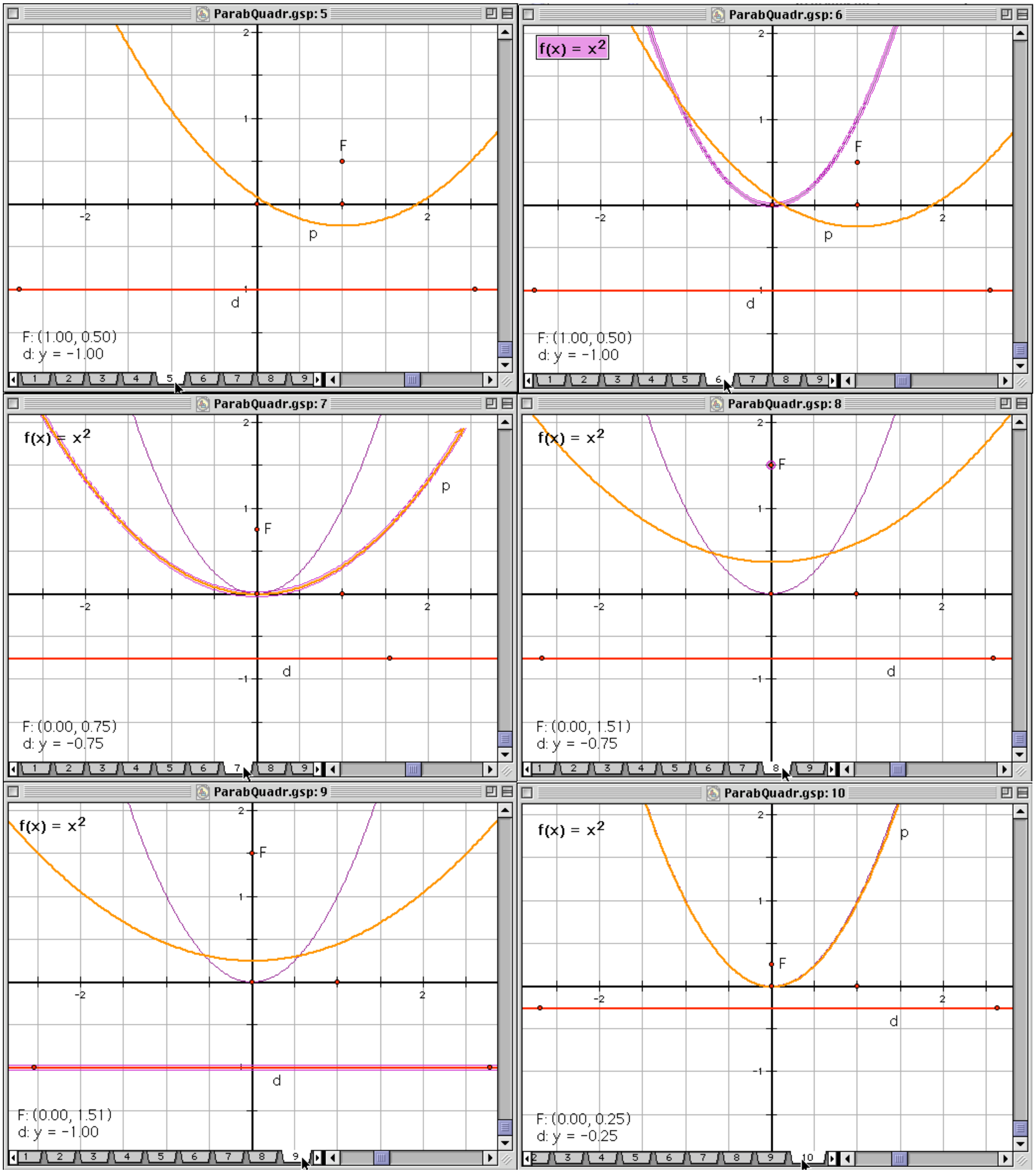
The parabola is introduced as the locus of a moving point equidistant from a fixed focus point and a fixed directrix line. Dragging a defining point on the directrix produces movement of the target point along its locus, leaving a trace behind, eventually forming a parabola (Figure 6.2.1-2).



Figures 6.2.1-4: Construction and manipulation of a parabola tool.

This construction can be adapted so that rather than the moving point being shown and traced, the locus is constructed directly. In effect, then, the parabola becomes an object which can be modified by manipulating the position of focus and directrix (Figures 6.2.3-4).

This tool can now be extended to display the positioning of focus and directrix in coordinate terms (Figure 6.2.5), and applied to the analysis of graphs (Figure 6.2.6). In this example, the quadratic graph to be analysed will prove to be parabolic, but the tool would be equally valuable in revealing that a curved graph was not actually parabolic. (And, of course, similar hyperbolic, circular, elliptic tools and the like could be devised). The screensnaps sketch a sequence of manipulations in which the tool is dragged into a suitable initial position respecting apparent symmetries (Figure 6.2.7), the effects of dragging focus and directrix are ascertained (Figures 6.2.8-9). and the position of fit between parabola tool and quadratic graph is found (Figure 6.2.10).



Figures 6.2.5-10: Fitting a parabola tool to a quadratic graph.

Once a good fit has been found, it can be analysed algebraically. In the example shown, the focus appears to be (0, 1/4) and the directrix y = -1/4. Using the locus definition of a parabola, an appropriate equation can be stated and transformed as follows:

$$(y + 1/4)^2 = (y - 1/4)^2 + x^2$$

$$y^2 + (1/2)y + (1/16) = y^2 - (1/2)y + (1/16) + x^2$$

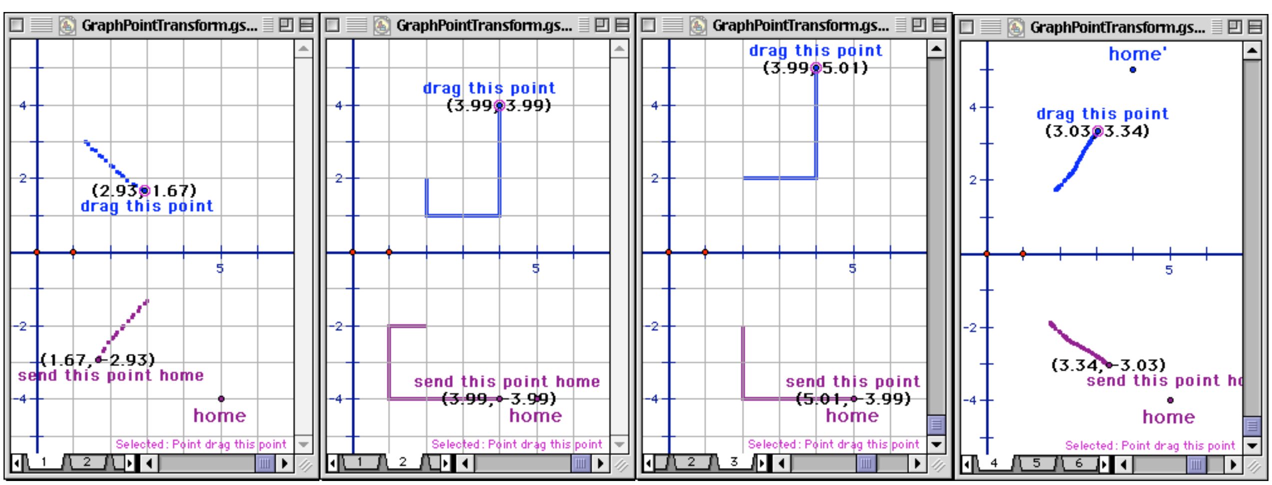
$$y = x^2$$

7 Transformational properties of graphs

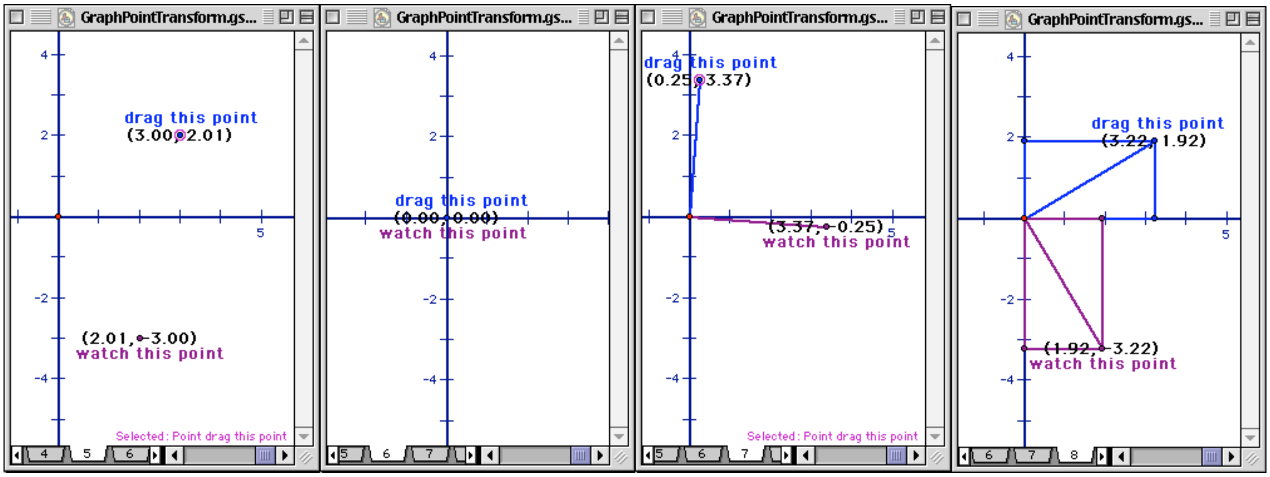
As illustrated in the previous section, graphs can be treated as geometric objects in their own right. Consequently, they are open to analysis in transformational terms, as well as classical and coordinate. Indeed, transformational approaches offer a powerful way of (re)framing many of the important properties of graphs. Again, access to graph plotting software has encouraged attention to such ideas, but conventional graph plotters only permit transformation of graphs to be investigated indirectly through inference from the effects of changing parameters in their equations. With graph plotting DGS, it becomes possible to treat such issues in terms of a more complete and coherent combined system of geometrical and algebraic operations.

7.1 Transformation of graphed points

Developing an intuitive sense of different types of transformation, and formalising this in terms of geometric and algebraic concepts can be assisted by activities in which such properties are explored informally and then analysed more formally. Figures 7.1.1-4 show an obvious way in which a DGS template can be configured to show a point and its image under an unknown transformation. Here the task has been framed as one of sending the image to its 'home' by dragging the defining point appropriately. The software is set to show traces of the movement of both points. A very natural initial strategy is to drag the defining point in the direction that takes it and/or the image towards 'home'; in such a case, the results are surprising (Figure 7.1.1). This encourages investigation of the relationship between the direction of dragging and the direction of movement of the image point: this can be approached by using the grid lines to guide the dragging movement, and varying direction systematically (Figure 7.1.2). Using these findings, it is then possible to work out an effective way of dragging along the grid lines (Figure 7.1.3). An alternative strategy depends on framing the problem quite differently: in terms of finding the unknown home' to which the defining point must be dragged in order to send its image correspondingly home (Figure 7.1.4). In particular, hiding the coordinate grid is likely to favour the development of this type of strategy. One way of identifying this home' is simply to reach it by some means or other. But a more satisfying way is to predict its position by analysing the general relationship between the coordinates of the defining point and its image, something which can be readily inferred from a relatively small number of cases.



Figures 7.1.1-4: Strategies for 'sending' a transformed point 'home'.



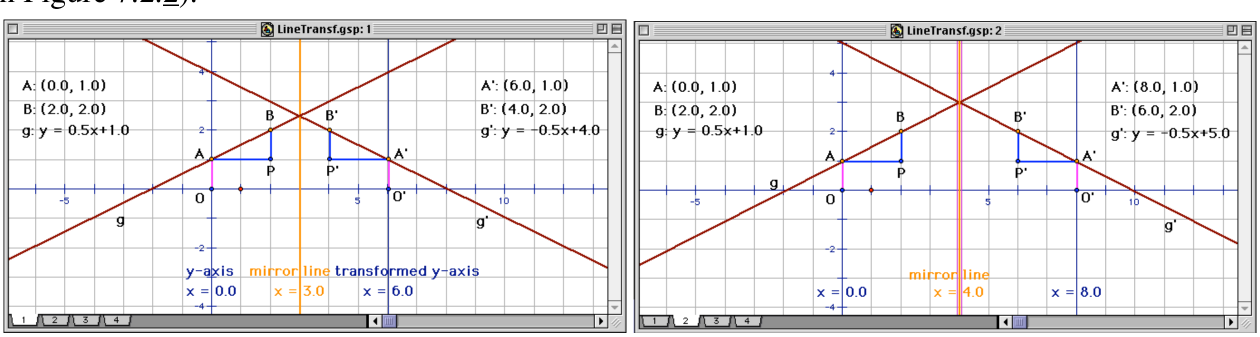
Figures 7.1.5-8: Strategies for analysing the transformation acting on a point.

A more formal counterpart of this activity simply sets the task of analysing the relationship between the defining point and its image (Figure 7.1.5), a relationship which can be conceived in both geometric and algebraic terms. Eyeballing the displayed coordinates is the obvious numeric-algebraic strategy. Geometrically, one tactic is to find where defining and image points coincide: in this case, only at the origin (Figure 7.1.6). This suggests constructing the segments between the origin and each of the points (Figure 7.1.7), and exploring the relationship in these terms. An ensuing tactic is to find special positions of this configuration by dragging (as suggested in Figure 7.1.7), suggesting that the two segments remain perpendicular. A stronger unifying strategy is to construct the rectangles formed by dropping perpendiculars from the two points to both the axes. This provides the means for a fuller geometrical analysis in terms of the four congruent triangles created, linked to an algebraic analysis in which the sides of each rectangle are directly related to the coordinates of the respective point (Figure 7.1.8).

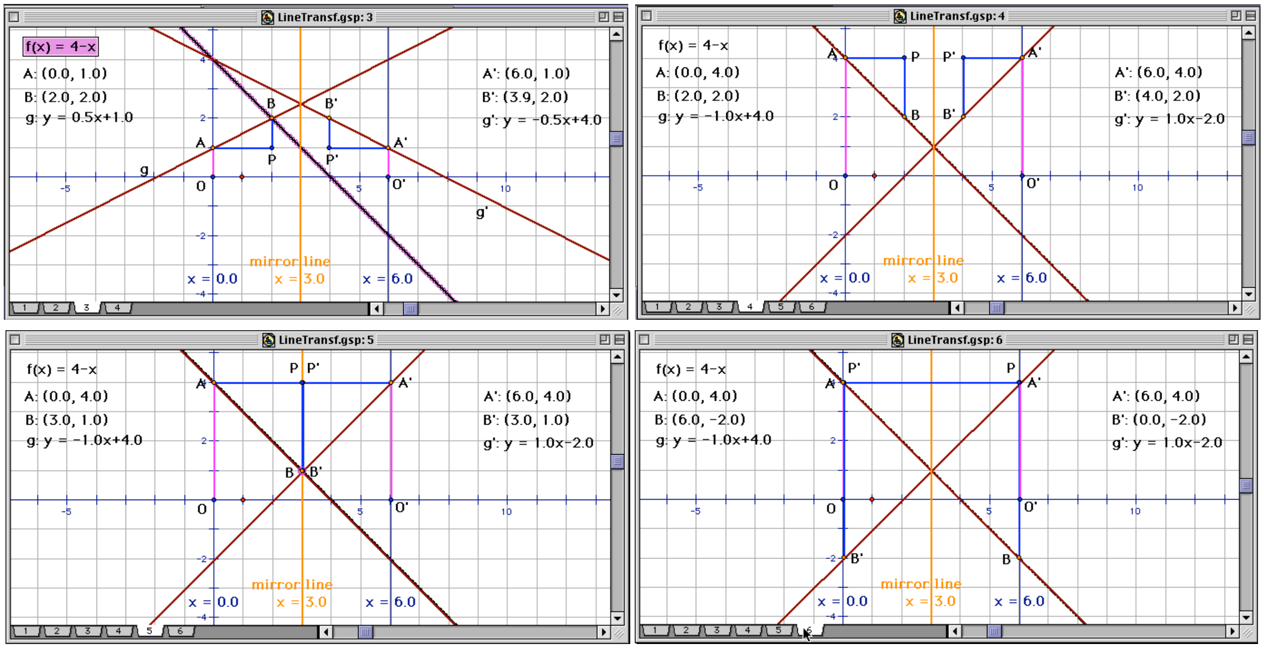
7.2 Transformation of line graphs

This example shows how the tool already used to analyse line graphs can be adapted so as to help analyse the effects of transformations on such graphs.

This is illustrated through the example of reflections in a vertical mirror line. The basic intercept-gradient-tool is adapted by construction of its reflected image (including the y-axis) in a vertical mirror line (as shown in Figure 7.2.1). The mirror line can be repositioned by dragging (as shown in Figure 7.2.2).



Figures 7.2.1-2: Development of a geometric tool to analyse reflected graphs.



Figures 7.2.3-6: Application of a geometric tool to analyse reflected graphs.

Once a linear function has been graphed (as shown in Figure 3.3.2.3) the original intercept-gradient-tool can be positioned on the graph (as shown in Figure 3.3.2.4), so that the reflection of the tool now indicates the position –and equation- that a reflection of the graph would take[12]. By dragging point B on the tool to the position where the graph cuts the mirror line, so that B and B', and P and P' coincide on the mirror line (as shown in Figure 3.3.2.5) the relationship between the gradients of the graph and its reflection becomes clearer: opposed horizontal shifts produce identical vertical shifts – and hence one gradient is the negative of the other. Similarly, by dragging point B on the tool to the position where the graph cuts the reflected y-axis, so that B' coincides with the intercept of the reflected graph on the y-axis, P coincides with A' and P' with A (as shown in Figure 3.3.2.6) the relationship between the intercepts of the graph and its reflection becomes clearer: the intercept of the reflected graph and the intersect of the original graph with the reflected axis have the same vertical position – which can be further related to the gradient of the graph and the position of the mirror line. Again, the main goal of working with this tool system is to establish generic schemes concerning the relationship between a line graph and its reflection.

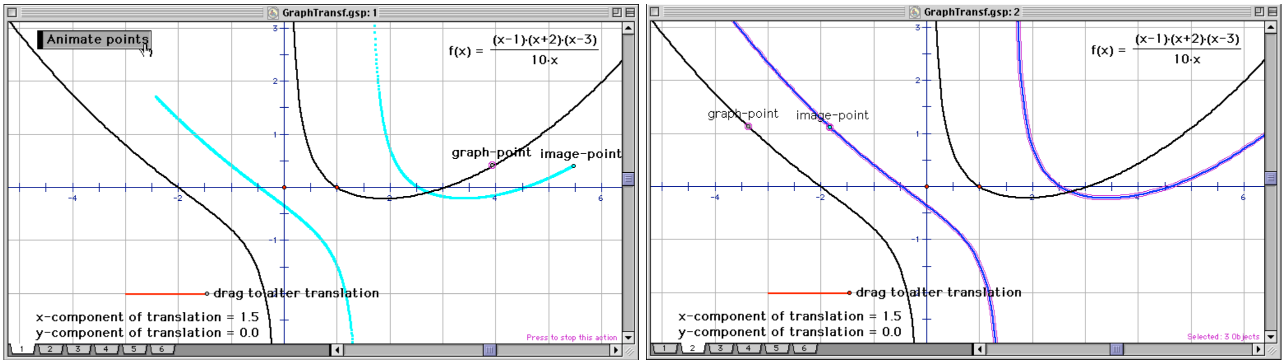
7.3 Transformation of function graphs

This final section focuses on a more generic approach to analysing the effects of transformations on function graphs. The example of translation will be used, but the approach is wholly generalisable.

The worksheet shown in Figure 7.3.1 creates a point free to move on the graph (defined by the displayed formula which can be edited to any other acceptable expression) and plots its image under a translation (defined by the segment at the foot of the sheet which can be modified by dragging the endpoint). On this worksheet, the movement of the free point on the graph has been

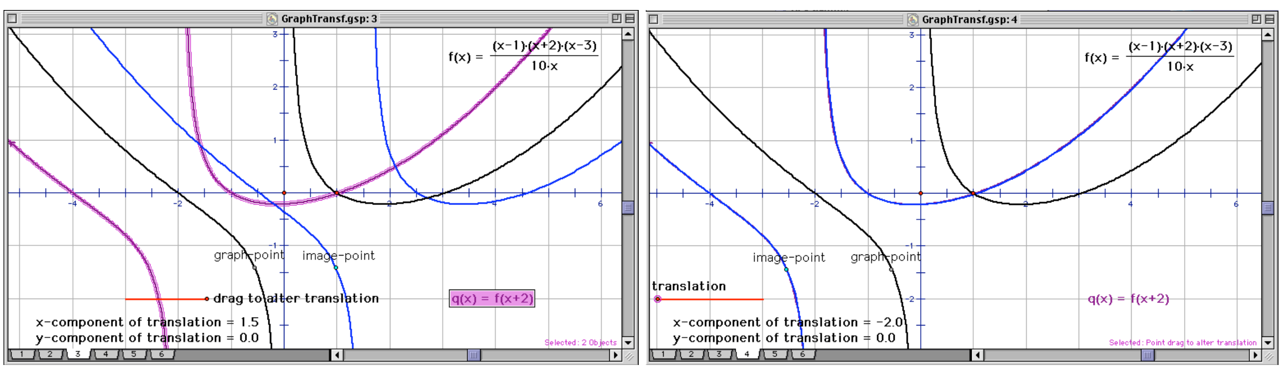
[12] Note that the function corresponding to the reflected line is not itself graphed, although the tool provides its equation. This means that properties of the reflected graph have to be deduced through appropriate positioning of the tool on the original graph.

animated, and the path of its image point traced, so producing the transformed graph of the function. In the worksheet shown in Figure 7.3.2, animation and trace have been removed; instead the image of the graph has been constructed directly as a geometrical object in its own right.

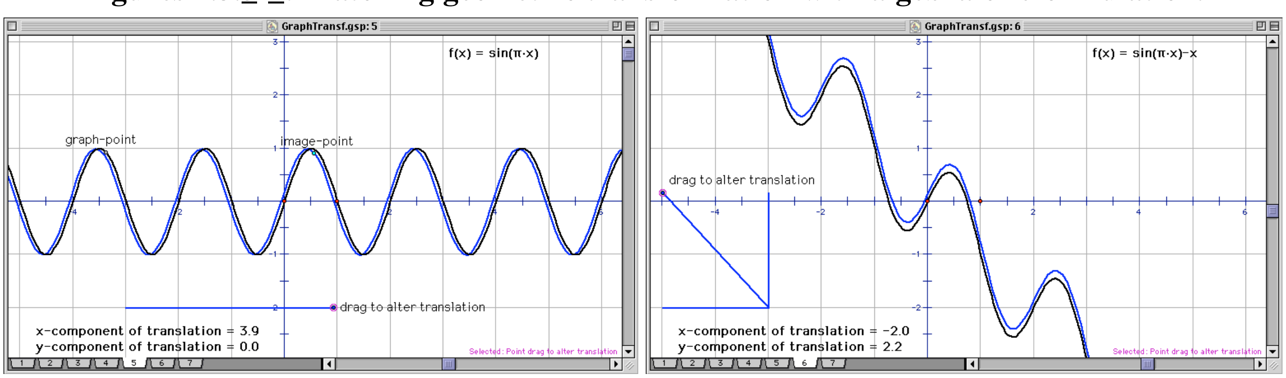


Figures 7.3.1-2: Basic templates for investigating and analysing translated graphs.

There are various ways in which the possibility of superimposing a further function graph can be exploited. One is to view the constructed locus and then attempt to fit a new function to it. Another is to predict the new function before viewing the locus, and then see if these fit. Equally, this new function can be defined in different ways; either by an independent expression; or by one dependent on the original function, such as f(x+2). There are, then, a number of interesting variants of this task, each of which has particular didactical characteristics. In Figure 7.3.3, a translation of +2 parallel to the x-axis has been fallaciously associated with the graph of y=f(x+2). In Figure 7.3.4, the translation has been reversed to produce a fit between new function and transformed graph, so correctly associating a translation of –2 with y=f(x+2). Other interesting tasks include trying to fit transformed graphs to their originals (as in the examples in Figure 7.3.5-6).



Figures 7.3.3-4: Matching geometric transformation with algebraic reformulation.



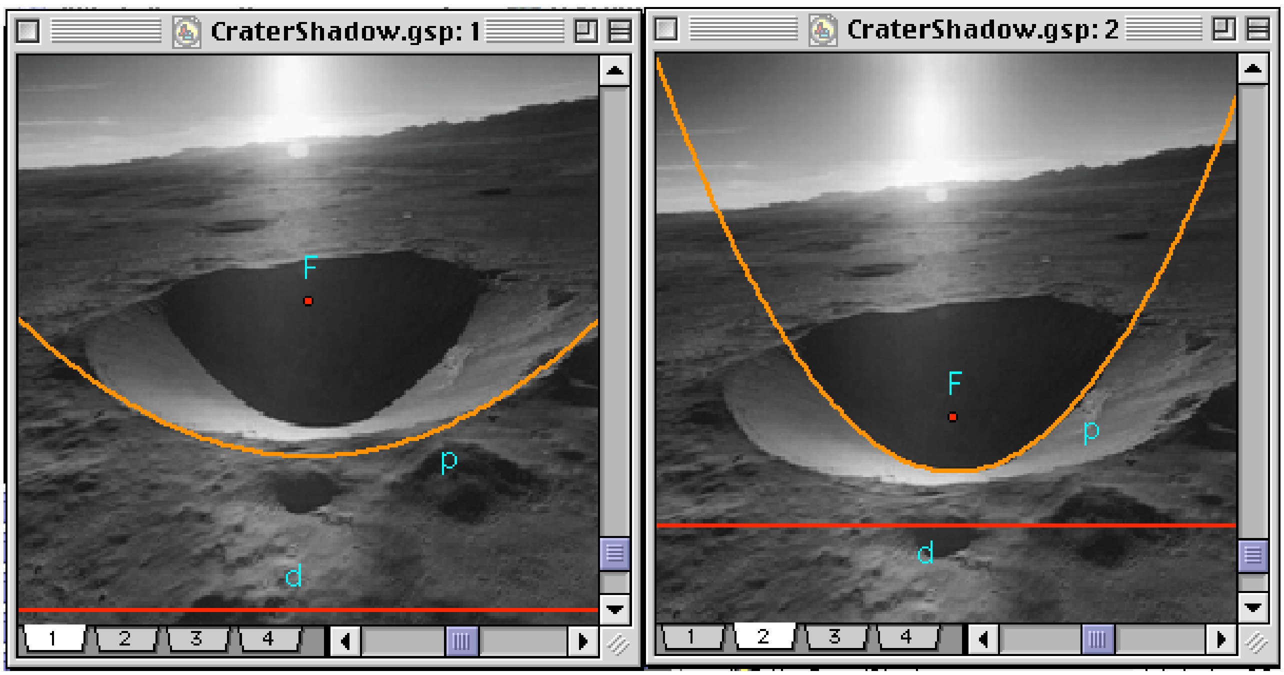
Figures 7.3.5-6: Transforming graphs to match themselves.

8 Geometric and algebraic modelling of visual images

Historically, the physical sciences provided the modelling problems which stimulated the synthesis of algebraic and geometric ideas and methods to develop tools for the analysis of covariation. The capacity to import photo or video images of physical pheomena into a DGS opens up new possibilities of exploiting this key connection. This will be illustrated by using the parabola tool introduced in §6 to analyse several images of this type.

8.1 Modelling a crater shadow

To provide, first, an example of purely geometric modelling. Figure 8.1.1 shows a photograph of a crater at the atomic test site in Nevada[13], which has been imported into a GSP file as a fixed background onto which the parabola tool has been superimposed. Dragging focus and directrix in a process of trial and improvement suggests that a parabola cannot be fitted to the rim of the crater (Figure 8.1.1) but can be fitted to the outline of the shadow (Figure 8.1.2). An extension is to explore the relationship between the axis of the parabola and the lightray in the photograph.



Figures 8.1.1-2: Fitting the parabola tool to a crater shadow.

8.2 Modelling a suspension bridge

Bridges provide a valuable source of lines and curves for geometric and algebraic modelling (Oldknow, 2002). Figure 8.2.1 shows a photograph of the Clifton suspension bridge[14], which has been imported into a GSP file as a fixed background. Figure 8.2.2 shows the result of copying and pasting the parabola tool into the window, and then dragging focus and directrix so as to fit the parabola to the main cable of the bridge through a process of trial and improvement.

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[13] Permission sought for reproduction [Found at http://www.parabola.org/magazine/current.html.]

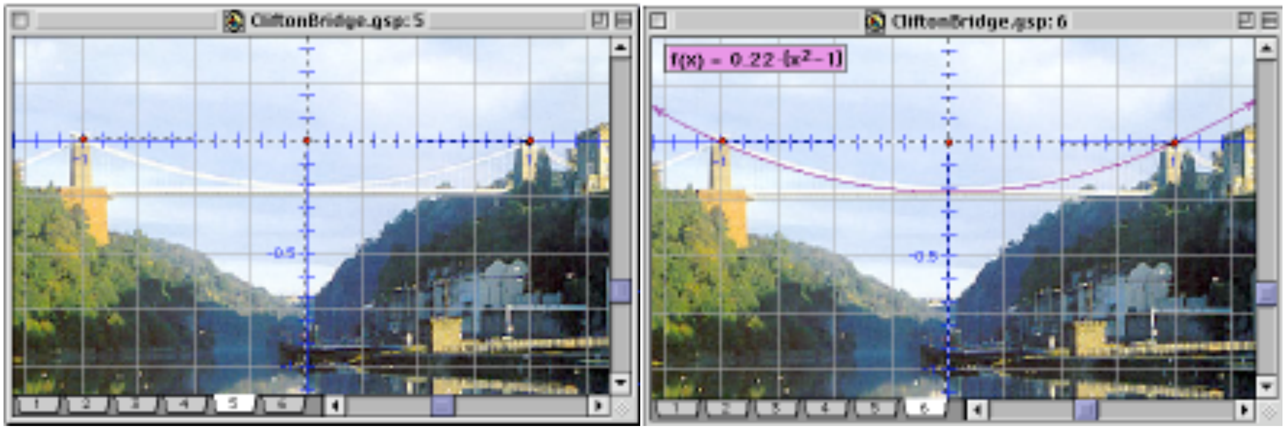
[14] Permission sought for reproduction [Found at http://www.clifton-suspension-bridge.org.uk/.]

Figures 8.2.1-2: Fitting the parabola tool to the bridge cable.

However, the fitting process can be guided geometrically by first constructing a line segment linking the highest points of the bridge towers, then the perpendicular to the segment passing through its midpoint, so indicating the axis of symmetry of the bridge and its main cable (Figure 8.2.3). The focus of the parabola tool can be merged to lie on this axis, as considerations of symmetry suggest that it must. But because they have been independently constructed, it is not technically possible to likewise constrain the directrix to be perpendicular to the axis. However, constructing the perpendicular to the directrix through its intersection with the axis provides an indicator for this criterion (Figure 8.2.4). Alternatively, the parabola tool can be reconstructed, incorporating the additional constraints of focus lying on axis and directrix perpendicular to axis.

Figures 8.2.3-4: Analysing the fitting of the parabola tool to the bridge cable.

Similarly, the constructed lines indicate one suitable position for coordinate axes, allowing a corresponding algebraic expression to be fitted to the main cable. Again, while this task can be approached through trial and improvement, it is also possible to capitalise on algebraic analysis to greatly simplify the problem. In the approach shown in Figure 8.2.5, the half-span of the bridge has been taken as the unit measure, so that the tops of the bridge towers lie at positions (-1, 0) and (1, 0). A quadratic polynomial passing through these points must take the form $c(x + 1)(x - 1)$. Reading off the minimum value of the polynomial at around (0, -0.22) indicates the value of c required, allowing the expression to be graphed to be deduced directly without any trialling (Figure 8.2.6). There are further variants if alternative positions for the axes are considered.



Figures 8.2.5-6: Fitting an algebraic graph to the cable .

8.3 Modelling a water jet

A further example, offering similar potential for analysis, is a jet of water from a drinking fountain[15]. Here, it is particularly interesting that the parabola appears to provide a better model for the jet as it rises from the nozzle than as it then falls away. (Figure 8.3.1).

Figure 8.3.1: Fitting the parabola tool to a water jet.

[15] Reproduced with the permission of Dr F Wattenberg and the Journal of Online Mathematics and Its Applications . [Found at http://www.joma.org/vol2/articles/wattenberg/JOMA_article/water_fountain.html].

9 Acknowledgements

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